RANKS OF DIVISORS ON HYPERELLIPTIC CURVES AND GRAPHS UNDER SPECIALIZATION

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Abstract. Let (G, ω) be a hyperelliptic vertex-weighted graph of genus $g \geq 2$. We give the characterization of (G, ω) for which there exists a smooth projective curve X of genus g over a complete discrete valuation field with the reduction graph (G, ω) such that the ranks of any divisors are preserved under specialization. We also explain, for a given vertex-weighted graph (G, ω) in general, how the existence of such X relates the Riemann–Roch formulae for X and (G, ω).

1. Introduction and statements of the main results

1.1. **Introduction.** The theory of divisors on smooth projective curves has been actively and deeply studied since the nineteenth century (cf. [4, 5]). It has been found that, also on graphs, there exists good theory of divisors (such as linear systems, linear equivalences, canonical divisors, degrees, and ranks). A Riemann–Roch formula, one of the most important formulae in the theory of divisors, was established by Baker–Norine on finite loopless graphs in their foundational paper [7]. A Riemann–Roch formula on tropical curves was independently proven by Gathmann–Kerber [15] and Mikhalkin–Zharkov [20]. Further, a Riemann–Roch formula on vertex-weighted graphs was proven by Amini–Caporaso [3], and on metrized complexes by Amini–Baker [1].

As Baker [6] revealed, the above similarity between the theory of divisors on curves and that on graphs is not just an analogy. Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k. Let X be a geometrically irreducible smooth projective curve over $\mathbb{K}$. An R-curve means an integral scheme of dimension 2 that is projective and flat over $\mathrm{Spec}(R)$. A *semi-stable model* of X is an R-curve $\mathscr{X}$ whose generic fiber is isomorphic to X and whose special fiber is a reduced scheme with at most nodes as singularities. For simplicity, suppose that there exists a semi-stable model $\mathscr{X}$ of X over $\mathrm{Spec}(R)$. (In general, we replace $\mathbb{K}$ by a finite extension $\mathbb{K}'$.) Let (G, ω) be the (vertex-weighted) reduction graph of $\mathscr{X}$, where G is the dual graph of the special fiber of $\mathscr{X}$ with natural vertex-weight function ω on G. Let Γ be the metric graph associated to G, where each edge is assigned length 1 (see §2 for details). To a point $P \in X(\mathbb{K})$, one can naturally associate a vertex v of G. This assignment is called the specialization map, and extends to $\tau : X(\overline{\mathbb{K}}) \to \Gamma_{\mathbb{Q}}$, where $\overline{\mathbb{K}}$ is a fixed algebraic closure of $\mathbb{K}$ and $\Gamma_{\mathbb{Q}}$ is the set of points on Γ whose distance from every vertex of G is rational. Let $\tau_* : \mathrm{Div}(X_{\overline{\mathbb{K}}}) \to \mathrm{Div}(\Gamma_{\mathbb{Q}})$ be the induced map on divisors, and let r_X (resp. r_Γ, $r_{(\Gamma,\omega)}$) denotes the rank of divisors on X (resp. Γ, (Γ, ω)) (see §2 for details). In [6], Baker showed $r_\Gamma(\tau_*(\widetilde{D})) \geq r_X(\widetilde{D})$ for any $\widetilde{D} \in \mathrm{Div}(X_{\mathbb{K}})$, a result now called Baker's Specialization Lemma. Amini–Caporaso [3] showed the specialization lemma for vertex-weighted graphs, and Amini–Baker [1] for metrized complexes.

This interplay between curves and graphs, especially the specialization lemma, has yielded several applications to the classical algebraic geometry such as a tropical proof of the famous Brill–Noether theorem [13] (see also [10], [19]). We remark that (Γ, ω) is seen as the Berkovich skeleton of X, and that further progress is under way within a framework of the Berkovich analytic space and the tropical variety of an algebraic variety over $\mathbb{K}$ (see, for example, [2], [9], [16], [21]).

In this paper, we study when the ranks of divisors are preserved under the specialization map (see Proposition 1.4 for our original motivation). By a finite graph, we mean an unweighted, finite

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connected multigraph. (This terminology is slightly different from [7], as we allow the existence of loops.) A vertex-weighted graph (G, ω) is a pair of a finite graph G and a function $w : V(G) \to \mathbb{Z}_{\geq 0}$, where $V(G)$ denotes the set of vertices of G.

Question 1.1. Let (G, ω) be a vertex-weighted graph, and let Γ be the metric graph associated to G. Under what condition on (G, ω), does there exist a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G, ω) satisfying the following condition?

- (C) Let X the generic fiber of $\mathscr{X}$, and $\tau : X(\overline{\mathbb{K}}) \to \Gamma_{\mathbb{Q}}$ be the specialization map. Then, for any $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, there exists a divisor $\widetilde{D} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $D = \tau_*(\widetilde{D})$ and $r_{(\Gamma,\omega)}(D) = r_X(\widetilde{D})$.

The purpose of this paper is to answer Question 1.1 for *hyperelliptic* graphs. Here, a vertex-weighted graph (G, ω) is hyperelliptic if the genus of (G, ω) is at least 2 and there exists a divisor D on Γ such that $\deg(D) = 2$ and $r_{(\Gamma,\omega)}(D) = 1$ (see Definition 3.10). An edge e of G is called a *bridge* if deletion of e makes G disconnected. Let G_1, G_2 denote the connected components of $G \setminus \{e\}$, which are equipped with the vertex-weight functions ω_1, ω_2 given by the restriction of ω. A bridge is called a *positive-type* bridge if both (G_1, ω_1) and (G_2, ω_2) have genus at least 1.

Under the notation in Question 1.1, we also consider the following condition (C'), which implies (C) (see Lemma 8.2).

- (C') For any $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, there exist a divisor $E = \sum_{i=1}^{k} n_i[v_i] \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ that is linearly equivalent to D and a divisor $\widetilde{E} = \sum_{i=1}^{k} n_i P_i \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $\tau(P_i) = v_i$ for any $1 \leq i \leq k$ and $r_{(\Gamma,\omega)}(E) = r_X(\widetilde{E})$.

Our main result is as follows.

Theorem 1.2. *Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k. Assume that* $\mathrm{char}(k) \neq 2$. *Let (G, ω) be a hyperelliptic vertex-weighted graph. Then the following are equivalent.*

- (i) *Every vertex v of G has at most $(2\,\omega(v) + 2)$ positive-type bridges emanating from v.*
- (ii) *There exists a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G, ω) which satisfies the condition* (C).
- (iii) *There exists a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G, ω) which satisfies the condition* (C').

As a corollary, we have the following vertex-weightless version. An semi-stable R-curve $\mathscr{X}$ is said to be *strongly semi-stable* if every component of the special fiber is smooth, and *totally degenerate* if every component of the special fiber is a rational curve. Let (G, ω) be the vertex-weighted reduction graph of an R-curve $\mathscr{X}$. Note that, if $\mathscr{X}$ is strongly semi-stable, then G is loopless, and if $\mathscr{X}$ is totally degenerate, then $\omega = \mathbf{0}$.

Corollary 1.3. *Let $\mathbb{K}, R, k$ be as in Theorem 1.2. Let $G = (G, \mathbf{0})$ be a loopless hyperelliptic graph. Then the following are equivalent.*

- (i) *Every vertex of G has at most 2 positive-type bridges emanating from it.*
- (ii) *There exists a regular, strongly semi-stable, totally degenerate R-curve $\mathscr{X}$ with reduction graph G which satisfies the condition* (C) *(with r_Γ in place of $r_{(\Gamma,\omega)}$).*
- (iii) *There exists a regular, strongly semi-stable, totally degenerate R-curve $\mathscr{X}$ with reduction graph G which satisfies the condition* (C') *(with r_Γ in place of $r_{(\Gamma,\omega)}$).*

We have come to consider Question 1.1 in our desire to understand relations between Riemann–Roch formulae on graphs and those on curves. Indeed, we have the following. (Since the Riemann–Roch formula for vertex-weighted graphs is a corollary of that for vertex-weightless graphs, we show the vertex-weightless version.)

Proposition 1.4. *Let G be a finite loopless graph and Γ the metric graph associated to G. Assume that there exist a complete discrete valuation field $\mathbb{K}$ with the ring integers R, and a regular, strongly*

semi-stable, totally degenerate R-curve $\mathscr{X}$ with reduction graph G which satisfies the condition (C)*. Then the Riemann–Roch formula on $\Gamma_{\mathbb{Q}}$ is deduced from the Riemann–Roch formula on $X_{\overline{\mathbb{K}}}$, where X is the generic fiber of $\mathscr{X}$.*

Let G be a loopless hyperelliptic graph. Let $\overline{G}$ be the hyperelliptic graph that is obtained by contracting all the bridges of G. Then Corollary 1.3, Proposition 1.4 and comparison of divisors on G and $\overline{G}$ give a new proof of the Riemann–Roch formula on a loopless hyperelliptic graph G (see Remark 8.5). It should be noted, however, that this proof still much uses combinatorics of the divisor classes on hyperelliptic graphs, as Theorem 1.2 is obtained by studying reduced divisors.

Let (G, ω) be a vertex-weighted graph, and Γ the metric graph associated to G. Question 1.1 is also of interest from the viewpoint of the Brill-Noether theory: For fixed integers $d, r \geq 0$, we put $W_d^r(\Gamma_{\mathbb{Q}}, w) := \{D \in \mathrm{Div}(\Gamma_{\mathbb{Q}}) \mid \deg(D) = d, r_{(\Gamma,\omega)}(D) \ \geq r\}$; If the answer to Question 1.1 is true with an R-curve $\mathscr{X}$ with generic fiber X, then we will have $\tau_*(W_d^r(X_{\overline{\mathbb{K}}})) = W_d^r(\Gamma_{\mathbb{Q}}, w)$.

1.2. **Remarks.** A number of remarks are in order.

Remark 1.5. In this paper, we consider vertex-weighted graphs (i.e., not only vertex-weightless finite graphs), since vertex-weighted graphs appear naturally in tropical geometry and Berkovich analytic spaces. We also show that the answer to Question 1.1 it true for any vertex-weighted graph of genus 0 or 1 (see Proposition 8.4).

Remark 1.6. The main part of the proof of Theorem 1.2 is to show that (i) implies (iii). To this end, we obtain several results in the theory of divisors on hyperelliptic metric graphs as follows (See §1.3 below):

- Properties of reduced divisors on hyperelliptic metric graphs (Theorem 1.10);
- A formula for the ranks of divisors on hyperelliptic metric graphs (Theorem 1.11).

Remark 1.7. The condition (C') is in general *not* equivalent to the following condition:

(C") For any $D = \sum_{i=1}^k n_i[v_i] \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, there exist $P_i \in X(\overline{\mathbb{K}})$ with $\tau(P_i) = v_i$ for any $1 \leq i \leq k$ such that $r_{(\Gamma,\omega)}(D) = r_X(\sum_{i=1}^k n_i P_i)$.

See Example 8.8. This example is interesting in two senses. First, by the condition (C), there always exists $\widetilde{D} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ with $\tau_*(\widetilde{D}) = D$ and $r_{(\Gamma,\omega)}(D) = r_X(\widetilde{D})$, but this example shows that $\widetilde{D}$ is not simply of the form $\sum_{i=1}^k n_{v_i} P_i$ with $\tau(P_i) = v_i$. Secondly, by the condition (C'), if we replace D by a divisor $E = \sum_{j=1}^{\ell} m_j[w_j]$ with $E \sim D$, then we can indeed lift E in X as a simple form $\widetilde{E} = \sum_{j=1}^{\ell} m_j Q_j$ with $\tau(Q_j) = w_j$ preserving the ranks $r_{(\Gamma,\omega)}(E) = r_X(\widetilde{E})$.

Remark 1.8. In a very recent paper [2], Amini–Baker–Brugallé–Rabinoff have studied lifting of harmonic morphisms of metrized complexes, among others, to morphisms of algebraic curves (see also Theorem 1.9 below). In [2, §10.11], they have discussed lifting divisors of given rank, giving several examples for which various specialization lemmas do not attain the equality. Question 1.1 will be interesting from this perspective, and Theorem 1.2 gives a clean picture in the case of hyperelliptic graphs. We remark that the problem of lifting divisors that preserve the ranks is studied by Caporaso [11] between connected reduced curves and associated graphs. We also remark that Cools, Draisma, Payne and Robeva considered a certain graph $G_\circ$ of g loops to give a tropical proof of the Brill–Noether theorem and that their conjecture [13, Conjecture 1.5] concerns lifting of divisors that preserves the ranks between $G_\circ$ and a regular, strongly semi-stable, totally degenerate R-curve with reduction graph $G_\circ$.

1.3. **Strategy of the proof and other results.** We now explain our strategy to prove Theorem 1.2. Our starting point is the following theorem.

Theorem 1.9 (cf. [2, Theorem 1.10])**.** *Let $\mathbb{K}, R, k$ be as in Theorem 1.2, and let (G, ω) be a vertex-weighted hyperelliptic graph. Then the condition* (i) *of Theorem 1.2 is equivalent to the existence of a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G, ω) such that $\mathscr{X}_{\mathbb{K}}$ is hyperelliptic.*

Theorem 1.9 has very recently been proven by Amini–Baker–Brugallé–Rabinoff [2, Theorem 1.10] as a corollary of their deep studies of canonical gluing and star analytic spaces (during the preparation of this paper). We will give an independent proof of Theorem 1.9 using equivariant deformation, which is also a natural way to prove Theorem 1.9 (see Theorem 4.6). Theorem 1.9 shows that (ii) implies (i) in Theorem 1.2. Since (C') is stronger than (C) (see Lemma 8.2), the condition (iii) implies (ii) in Theorem 1.2.

The main part of the proof of Theorem 1.2 is to show that (i) implies (iii). For a metric graph Γ and $v_0 \in \Gamma$, a divisor $D \in \mathrm{Div}(\Gamma)$ is said to be *v_0-reduced* if D is effective away from v_0 and satisfies several nice properties (see Definition 2.4). The notion of reduced divisors was introduced by Baker–Norine [7], and is a powerful tool in computing the ranks of divisors. We study reduced divisors on hyperelliptic metric graphs.

Theorem 1.10. *Let Γ be a hyperelliptic metric graph of genus g. We fix $v_0 \in \Gamma$ satisfying* (3.2)*. Let $D \in \mathrm{Div}(\Gamma)$ be a v_0-reduced divisor on Γ, and let $D(v_0)$ denote the coefficient of D at v_0. Then, if $\deg(D) - D(v_0) \le g-1$, then there exists $w \in \Gamma \setminus \{v_0\}$ such that $D + [w]$ is a v_0-reduced divisor.*

Let Γ be a hyperelliptic metric graph. We set, for an effective divisor $D \in \mathrm{Div}(\Gamma)$, $p_\Gamma(D) = \max\{r \in \mathbb{Z}_{\ge 0} \mid |D - 2r[v_0]| \ne \emptyset\}$. Let (G, ω) be a hyperelliptic vertex-weighted graph and Γ the metric graph associated to G. Let Γ^ω be the virtual weightless metric graph of (Γ, ω), and we define $p_{(\Gamma,\omega)}$ to be p_{Γ^ω} (see §3.4 for details). Using Theorem 1.10, we compute $r_{(\Gamma,\omega)}(D)$ in terms of $p_{(\Gamma,\omega)}(D)$, which is a key ingredient of the proof of Theorem 1.2.

Theorem 1.11. *Let (G, ω) be a hyperelliptic vertex-weighted graph of genus g and Γ the metric graph associated to G. Then, for any effective divisor D on Γ, we have*

$$r_{(\Gamma,\omega)}(D) = \begin{cases} p_{(\Gamma,\omega)}(D) & (\text{if } \deg(D) - p_{(\Gamma,\omega)}(D) \le g), \\ \deg(D) - g & (\text{if } \deg(D) - p_{(\Gamma,\omega)}(D) \ge g+1). \end{cases}$$

There is a corresponding formula in the classical setting of ranks of divisors on hyperelliptic curves (see Theorem 7.1). We deduce (iii) from (i) in Theorem 1.2, combining Theorem 1.9, Theorem 1.11, Theorem 7.1 and Baker's Specialization Lemma.

The organization of this paper is as follows. In Sect. 2, we briefly recall the theory of divisors on metric graphs. In Sect. 3, we consider hyperelliptic graphs. In Sect. 4, we consider hyperelliptic semi-stable curves and prove Theorem 1.9 using equivariant deformation. In Sect. 5, we study reduced divisors on a hyperelliptic graph, and prove Theorem 1.10. The proof of Theorem 1.10 is combinatorial. In Sect. 6, we study ranks of divisors on a hyperelliptic graph, and prove Theorem 1.11. In Sect. 7, we show a formula on ranks of divisor on hyperelliptic curves, analogous to Theorem 1.11. In Sect. 8, we prove Theorem 1.2 and Proposition 1.4. We also consider Question 1.1 for vertex-weighted graphs of genus 0 or 1. In the appendix, we put together some results on deformation theory which are needed in Sect. 4.

2. Preliminaries

In this section, we briefly recall the theory of divisors and a Riemann–Roch formula on a compact metric graph, Baker's specialization lemma, and the notion of reduced divisors on a metric graph. We also recall some properties of a vertex-weighted graph, and irreducible decomposition and contraction of metric graphs.

2.1. **Riemann–Roch formula on a metric graph.** We briefly recall the theory of divisors on metric graphs. We refer the reader to [7], [15], [18], [20] for details and further references.

Throughout this paper, a *finite graph* means an unweighted, finite connected multigraph. Notice that we allow the existence of loops. For a finite graph G, let $V(G)$ denotes the set of vertices, and $E(G)$ the set of edges. The genus of G is defined by $g(G) = |E(G)| - |V(G)| + 1$. For $v \in V(G)$, the *valence* $\mathrm{val}(v)$ of V is the number of edges emanating from v. Recall from the introduction that $e \in E(G)$ is called a *bridge* if deletion of e makes G disconnected. A vertex v of G is a *leaf end* if $\mathrm{val}(v) = 1$. A *leaf edge* is an edge of G that has a leaf end.

Remark 2.1. (1) Let G be a finite graph. Then the following are equivalent: (i) G has no vertices of valence 1; (ii) G has no leaf edges.
(2) A finite graph without bridges is also called a 2-edge-connected graph.

An *edge-weighted graph* (G, ℓ) is a pair of a finite graph G and a function (called a length function) $\ell : E(G) \to \mathbb{R}_{>0}$. In other words, an edge-weighted graph means a finite graph having each edge assigned a positive length. A compact connected metric graph Γ is the underlining metric space of an edge-weighted graph (G, ℓ). We say that (G, ℓ) is a model of Γ. There are many possible models for Γ. However, if Γ is not a circle, we can canonically construct a model $(G_\circ, \ell)$ of Γ as follows. The set of vertices is given by $V(G_\circ) := \{v \in \Gamma \mid \mathrm{val}(v) \neq 2\}$, and the set of edges $E(G_\circ)$ correspond to the connected component of $\Gamma \smallsetminus V(G_\circ)$. Since each connected component of $\Gamma \smallsetminus V(G_\circ)$ is an open interval, its length determines the length function ℓ. The model $(G_\circ, \ell)$ is called the *canonical model* of Γ.

Let Γ be a compact connected metric graph. By an edge of Γ, we mean an edge of the underlining graph $G_\circ$ of the canonical model $(G_\circ, \ell)$. Similarly, by a bridge (reps. a leaf edge) of Γ, we mean a bridge (reps. a leaf edge) of $G_\circ$. Let e be an edge of Γ, which is not a loop. We regard e as a closed subset of Γ (i.e. including the endpoints of e). The interior of e is denoted by $\mathring{e}$.

The genus $g(\Gamma)$ of a compact connected metric graph Γ is defined by its first Betti number, which is equal to $g(G)$ of any model (G, ℓ) of Γ. The elements of the free abelian group $\mathrm{Div}(\Gamma)$ generated by points of Γ is called *divisors* on Γ. For $D = \sum_{v \in \Gamma} n_v[v] \in \mathrm{Div}(\Gamma)$, its *degree* is defined by $\deg(D) = \sum_{v \in \Gamma} n_v$. We write the coefficient n_v at $[v]$ for $D(v)$. A divisor $D = \sum_{v \in \Gamma} n_v[v] \in \mathrm{Div}(\Gamma)$ is said to be *effective* if $n_v = D(v) \geq 0$ for any $v \in \Gamma$. If D is effective, we write $D \geq 0$.

A *rational function* on Γ is a piecewise linear function on Γ with integer slopes. We denote by $\mathrm{Rat}(\Gamma)$ the set of rational functions on Γ. For $f \in \mathrm{Rat}(\Gamma)$ and a point v in Γ, the sum of the incoming slopes of f at v is denoted by $\mathrm{ord}_v(f)$. This sum is 0 except for finitely many points of Γ, and one obtains

$$\mathrm{div}(f) := \sum_{v \in \Gamma} \mathrm{ord}_v(f)[v] \in \mathrm{Div}(\Gamma).$$

The set of *principal divisors* on Γ is defined by $\mathrm{Prin}(\Gamma) := \{\mathrm{div}(f) \mid f \in \mathrm{Rat}(\Gamma)\}$. Then $\mathrm{Prin}(\Gamma)$ is a subgroup of $\mathrm{Div}(\Gamma)$. Two divisors $D, E \in \mathrm{Div}(\Gamma)$ are said to be *linearly equivalent* if $D - E \in \mathrm{Prin}(\Gamma)$, and are denoted by $D \sim E$. For $D \in \mathrm{Div}(\Gamma)$, the complete linear series of $|D|$ is defined by

$$|D| = \{E \in \mathrm{Div}(\Gamma) \mid E \geq 0, \quad E \sim D\}.$$

The complete linear series $|D|$ has a structure of a polyhedral complex (see [15]).

Let G be a finite graph. We say that Γ is the metric graph associated to G if Γ is the underlining metric space of $(G, \mathbf{1})$, where $\mathbf{1}$ denotes the length function which assigns to each edge of G length 1. If this is the case, let $\Gamma_{\mathbb{Q}}$ denotes the set of points on Γ whose distance from every vertex of G is rational, and let $\mathrm{Div}(\Gamma_{\mathbb{Q}})$ denote the free abelian group generated by the elements of $\Gamma_{\mathbb{Q}}$.

Definition 2.2 (Rank of a divisor, cf. [7])**.** Let Γ be a compact connected metric graph. Let $D \in \mathrm{Div}(\Gamma)$. If $|D| = \emptyset$, then we set $r_\Gamma(D) := -1$. If $|D| \neq \emptyset$, we set

$$r_\Gamma(D) := \max \left\{ s \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{For any effective divisor } E \text{ with } \deg(E) = s, \\ \text{we have } |D - E| \neq \emptyset \end{array} \right\}.$$

For $v \in \Gamma$, the *valence* $\mathrm{val}(v)$ is the number of connected components in $U_v \smallsetminus \{v\}$, where U_v is any small neighborhood of v in Γ.

The *canonical divisor* of Γ is defined by $K_\Gamma := \sum_{v \in \Gamma} (\mathrm{val}(v) - 2)[v]$ (cf. [25]). It is easy to see $\mathrm{val}(v) - 2 = 0$ except for finitely many points v on Γ, so $K_\Gamma \in \mathrm{Div}(\Gamma)$.

Theorem 2.3 (Riemann–Roch formula on a metric graph, [7], [15], [20])**.** *Let Γ be a compact connected metric graph. For any $D \in \mathrm{Div}(\Gamma)$, one has*

$$r_\Gamma(D) - r_\Gamma(K_\Gamma - D) = \deg(D) + 1 - g(\Gamma).$$

2.2. **Reduced divisors on a metric graph.** We briefly review the notion of *reduced divisors* on a graph, which was introduced in [7] to prove the Riemann–Roch formula for a finite graph , and is a powerful tool in computing the rank of divisors.

Let Γ be a compact connected metric graph. For any closed subset A of Γ and $v \in \Gamma$, the *out-degree* of v from A, denoted by $\mathrm{outdeg}_A^{\Gamma}(v)$, is defined to be the maximum number of internally disjoint segments of $\Gamma \smallsetminus A$ with an open end v. Note that if $v \in A \smallsetminus \partial A$, then $\mathrm{outdeg}_A^{\Gamma}(v) = 0$. For $D \in \mathrm{Div}(\Gamma)$, a point $v \in \partial A$ is *non-saturated* with respect to A and D if $D(v) < \mathrm{outdeg}_A^{\Gamma}(v)$, and *saturated* otherwise.

Definition 2.4 (v_0-reduced divisor)**.** We fix a point $v_0 \in \Gamma$. A divisor $D \in \mathrm{Div}(\Gamma)$ is called a *v_0-reduced divisor* if D is non-negative on $\Gamma \smallsetminus \{v_0\}$, and every compact subset A of $\Gamma \smallsetminus \{v_0\}$ contains a non-saturated point $v \in \partial A$.

We remark that we may require that a compact subset A of $\Gamma \smallsetminus \{v_0\}$ be connected in the above definition. We put together useful properties of a v_0-reduced divisor in the following theorem:

Theorem 2.5 ([6], [7], [18])**.** *Let $D \in \mathrm{Div}(\Gamma)$ and $v_0 \in \Gamma$.*

1. *There exists a unique v_0-reduced divisor D_{v_0} that is linearly equivalent to D.*
2. *The divisor D is linearly equivalent to an effective divisor if and only if D_{v_0} is effective.*
3. *Suppose that Γ is the metric graph associated to a finite graph G. Then, if $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, then $D_{v_0} \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$.*

Lemma 2.6. *Let Γ be a compact connected metric graph, and e a bridge of Γ. Let $v_0 \in \Gamma$ be a point with $v_0 \notin \mathring{e}$. Let $D \in \mathrm{Div}(\Gamma)$ be a v_0-reduced divisor. Then $\mathrm{Supp}(D) \cap \mathring{e} = \emptyset$.*

Proof. Suppose that $w \in \mathrm{Supp}(D) \cap \mathring{e}$. Let Γ_1 be the connected component of $\Gamma \smallsetminus \{w\}$ that does not contain v_0. We set $A := \Gamma_1 \cup \{w\}$. Then A is a compact connected subset of $\Gamma \smallsetminus \{v_0\}$ and $\partial A = \{w\}$. Since $\mathrm{outdeg}_A^{\Gamma}(w) = 1$ and $D(w) \geq 1$, w is a saturated point for D. Since D is v_0-reduced, this is a contradiction. $\square$

2.3. **Specialization lemma.** In this subsection, following [6], we briefly recall the connection between linear systems on curves and those on graphs, and Baker's Specialization Lemma.

Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k. Let X be a geometrically irreducible smooth projective curve over $\mathbb{K}$. We assume that X has a *semi-stable* model over R, i.e., there exists a regular R-curve $\mathscr{X}$ whose generic fiber is isomorphic to X and whose special fiber $\mathscr{X}_0$ is a reduced scheme that has at most nodes (i.e. ordinary double points) as singularities.

We note that, for Theorem 1.2 and other theorems in this paper, for a given vertex-weighted graph (G, ω), we will construct a curve X over $\mathbb{K}$ which do have a semi-stable model over R, so that the assumption of the existence of a semi-stable model is not restrictive. (In general, by the semi-stable reduction theorem, if we take a finite extension field $\mathbb{L}$ of $\mathbb{K}$ with ring of integers S, then there exists a semi-stable model of $X_{\mathbb{L}} := X \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{L})$ over S.)

The *dual graph* G associated to $\mathscr{X}_0$ is defined as follows. Let $X_1, \ldots, X_r$ be the irreducible components of $\mathscr{X}_0$. Then G has vertices $v_1, \ldots, v_r$ which correspond to $X_1, \ldots, X_r$, respectively. Two vertices v_i, v_j $(i \neq j)$ of G are connected by a_{ij} edges if $\# X_i \cap X_j = a_{ij}$. A vertex v_i has b_i loops if $\#\mathrm{Sing}(X_i) = b_i$. We call the dual graph of $\mathscr{X}_0$ the *reduction graph* of the R-curve $\mathscr{X}$.

Let Γ be the metric graph associated to G, where each edge is assigned length 1. Let $P \in X(\mathbb{K})$. By the valuative criterion of properness , P gives the section Δ_P over R, which meets an component of the special fiber in its smooth locus. Let $v \in G$ be the vertex corresponding to this component. We denote by $\tau : X(\mathbb{K}) \to \Gamma$ the map which assigns P to v. Suppose $\mathbb{K}'$ is a finite extension field of $\mathbb{K}$, with ring of integers R'. Let $\mathscr{X}'$ be the minimal resolution of $\mathscr{X} \times_R R'$. Then the generic fiber of $\mathscr{X}'$ is $X \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{K}')$. Let G' be the dual graph of the special fiber of $\mathscr{X}'$. Let Γ' be a metric graph whose underlining graph is G', where each edge is assigned length $1/[\mathbb{K}' : \mathbb{K}]$. Then

Γ' is naturally isometric to Γ. We have the specialization map (again denoted by τ by slight abuse of notation)

$$\tau : X(\overline{\mathbb{K}}) \to \Gamma.$$

The specialization map τ induces the group homomorphism

$$\tau_* : \operatorname{Div}(X_{\overline{\mathbb{K}}}) \to \operatorname{Div}(\Gamma).$$

Proposition 2.7 ([6])**.** (1) *One has* $\operatorname{Image}(\tau) = \Gamma_{\mathbb{Q}}$ *and* $\operatorname{Image}(\tau_*) = \operatorname{Div}(\Gamma_{\mathbb{Q}})$.
(2) *The map* τ_* *respects the linear equivalence.*
(3) *One has* $\tau_*(K_X) \sim K_\Gamma$.
(4) *For any* $\widetilde{D} \in \operatorname{Div}(X_{\overline{\mathbb{K}}})$, $\deg \tau_*(\widetilde{D}) = \deg \widetilde{D}$.

Proof. For (1), see [6, Remark 2.3]. For (2), we refer to [6, Lemma 2.1]. For (3), we refer to [6, Lemma 4.19]. The statement (4) is obvious from the definition of τ. We note that, in [6], the each component of the special fiber $\mathscr{X}_0$ is assumed to be smooth. But the arguments in [6] also hold when a component of $\mathscr{X}_0$ has a node. □

We state Baker's Specialization Lemma [6]. Again, the arguments in [6] hold when a component of $\mathscr{X}_0$ has a node. (Notice that the rank of a divisor is measured by r_Γ, not by r_G.)

Theorem 2.8 (Baker's Specialization Lemma [6])**.** *For any* $\widetilde{D} \in \operatorname{Div}(X_{\overline{\mathbb{K}}})$, *one has* $r_\Gamma(\tau_*(\widetilde{D})) \geq r_X(\widetilde{D})$.

2.4. **Irreducible decomposition and contraction of a metric graph.** Let Γ be a compact connected metric graph. We recall irreducible decomposition $\Gamma_0 \vee \cdots \vee \Gamma_n$ of Γ and compare divisors on Γ and Γ_i. We will also compare divisors on a compact connected metric graph Γ and those on the metric graph obtained by contracting a bridge of Γ.

Irreducible decomposition. Let Γ be a compact connected metric graph. A point $v \in \Gamma$ is called a *cut-vertex* for Γ if v is a vertex of the canonical model of Γ and $\Gamma \smallsetminus \{v\}$ is disconnected. We say Γ to be *irreducible* if there is no cut-vertex for Γ. By convention, if Γ is a point, then Γ is not irreducible.

Let $\Gamma_0, \ldots, \Gamma_n$ be subgraphs of Γ, i.e., compact connected subsets of Γ equipped with the metric induced from Γ. We say that $\{\Gamma_0, \ldots, \Gamma_n\}$ is the set of *irreducible components* of Γ if it satisfies the following conditions:

(i) For $i = 0, \ldots, n$, Γ_i is an irreducible metric graph;
(ii) For $i \neq j$, $\Gamma_i \cap \Gamma_j$ either consists of a cut-vertex or is the empty set;
(iii) $\Gamma = \Gamma_0 \cup \cdots \cup \Gamma_n$.

If these conditions are met, we write $\Gamma = \Gamma_0 \vee \cdots \vee \Gamma_n$, and call it an *irreducible decomposition* of Γ. We note that the notion of irreducible decomposition is used in [24].

Let Γ be a compact connected metric graph that is not a point. By successively decomposing, we have a *unique* irreducible decomposition $\Gamma = \Gamma_0 \vee \cdots \vee \Gamma_n$.

For an effective divisor D on Γ, we define the restriction D_i of D to Γ_i to be the divisor on Γ_i given by $D_i(v) = D(v)$ for all $v \in \Gamma_i$.

Lemma 2.9. *Let* Γ *be a compact connected metric graph that is not a point, and let* $\Gamma = \Gamma_0 \vee \cdots \vee \Gamma_n$ *be the irreducible decomposition of* Γ. *We fix a point* $v_0 \in \Gamma_0$. *Let* D *be a divisor on* Γ. *For* $i = 1, \ldots, n$, *if* v_0 *is the cut-vertex of* Γ_0 *and* Γ_i, *that is, if* v_0 *is a cut-vertex of* Γ *with* $\Gamma_0 \cap \Gamma_i = \{v_0\}$, *then we let* $v_i := v_0$. *If* v_0 *is not the cut-vertex of* Γ_0 *and* Γ_i, *then we let* v_i *be the cut-vertex for* Γ *that disconnects* v_0 *from* $\Gamma_i \smallsetminus \{v_i\}$. *Let* D_i *be the restriction of* D *to* Γ_i. *Then,* D *is* v_0*-reduced if and only if* D_i *is* v_i*-reduced for any* $i = 0, \ldots, n$.

Proof. We first prove the “only if“ part. We fix any $i \in \{0, \ldots, n\}$. Let $A_i \subseteq \Gamma_i \smallsetminus \{v_i\}$ be a compact connected subset. We will show that D_i is v_i-reduced. We put

$$J_{A_i} = \{j \in \{0, \ldots, n\} \mid j \neq i, v_j \in A_i\},$$

and set $B_i = A_i \cup \bigcup_{j \in J_{A_i}} \Gamma_j$. Note that B_i is a compact connected subset of Γ.

If $v_i = v_0$, then $v_0 = v_i \notin B_i$. If $v_i \neq v_0$, then v_i disconnect v_0 from $\Gamma_i \smallsetminus \{v_i\}$, so that we also have $v_0 \notin B_i$. Since D is v_0-reduced, there is a non-saturated point $a \in \partial B_i$. Since ∂B_i in Γ coincides with ∂A_i in Γ_i, we have $a \in \partial A_i$. Further, we see $\mathrm{outdeg}_{A_i}^{\Gamma_i}(a) = \mathrm{outdeg}_{B_i}^{\Gamma}(a)$ by our definition of B_i, where $\mathrm{outdeg}^{\Gamma_i}$ and outdeg^{Γ} stand for the out-degrees in Γ_i and Γ, respectively. Then we have

$$D_i(a) = D(a) < \mathrm{outdeg}_{B_i}^{\Gamma}(a) = \mathrm{outdeg}_{A_i}^{\Gamma_i}(a),$$

so that a is a non-saturated point for D_i in Γ_i.

We show the "if" part. We take any compact connected subset $A \subseteq \Gamma \smallsetminus \{v_0\}$. There exists $i \in \{0, 1, \ldots, n\}$ such that $A_i := A \cap \Gamma_i \neq \emptyset$ and $v_i \notin A_i$. Since $v_i \notin A_i$ and ∂A_i in Γ_i is non-empty, there exists a non-saturated point $a_i \in \partial A_i$ for D_i. Noting that $a_i \in \partial A$ and $\mathrm{outdeg}_{A_i}^{\Gamma_i}(a_i) \leq \mathrm{outdeg}_{A}^{\Gamma}(a_i)$ by the definition of A_i, we obtain

$$D(a_i) = D_i(a_i) < \mathrm{outdeg}_{A_i}^{\Gamma_i}(a_i) \leq \mathrm{outdeg}_{A}^{\Gamma}(a_i).$$

It follows that a_i is also a non-saturated point for the divisor D on Γ. This completes the proof. □

Contraction. Let Γ be a compact connected metric graph. Suppose that Γ has a bridge e, and let Γ_1 be the graph obtained by contracting a bridge. Let $\rho_1 : \Gamma \to \Gamma_1$ be the retraction map.

Lemma 2.10 ([12, Lemma 3.11]). *Let Γ, Γ_1, ρ_1 be as above. Let $D \in \mathrm{Div}(\Gamma)$ and $D_1 \in \mathrm{Div}(\Gamma_1)$.*

(1) *We have $D \in \mathrm{Prin}(\Gamma)$ if and only if $\rho_{1*}(D) \in \mathrm{Prin}(\Gamma_1)$.*
(2) *We have $r_\Gamma(D) = r_{\Gamma_1}(\rho_{1*}(D))$.*
(3) *Suppose that the contracted bridge e is a leaf edge, so that we have a natural embedding $\jmath_1 : \Gamma_1 \hookrightarrow \Gamma$. Then we have $r_\Gamma(\jmath_{1*}(D_1)) = r_{\Gamma_1}(D_1)$.*

Proof. (1) See [12, Lemma 3.11]. (2) This follows from (1) by the argument in [8, Corollaries 5.10, 5.11]. (3) Since $\rho_{1*}(\jmath_{1*}(D_1)) = D_1$, the assertion follows from (2). □

2.5. **Vertex-weighted graph.** In this subsection, following [3], we briefly recall some properties of vertex-weighted graphs.

A *vertex-weighted* graph (G, ω) is a pair of a finite graph G and a function (called a vertex-weight function) $\omega : V(G) \to \mathbb{Z}_{\geq 0}$. The genus of (G, ω) is defined by $g(G, \omega) = g(G) + \sum_{v \in V(G)} \omega(v)$.

Let Γ be the metric graph associated to G (i.e., each edge of G is assigned length 1). The genus $g(\Gamma, \omega)$ of (Γ, ω) is defined by $g(\Gamma, \omega) = g(\Gamma) + \sum_{v \in V(G)} \omega(v)$, which coincides with $g(G, \omega)$.

Let (G, ω) be a vertex-weighted graph. For each vertex $v \in V(G)$, we add $\omega(v)$ loops to the vertex v to make a new finite graph G^ω. The graph G^ω is called the *virtual weightless finite graph* associated to a vertex-weighted graph (G, ω). The attached loops are called *virtual loops*. Let Γ^ω be the metric graph associated to G^ω (i.e., each edge of G^ω is assigned length 1). We call Γ^ω the *virtual weightless metric graph* associated to (G, ω). Then $g(G^\omega) = g(G, \omega) = g(\Gamma^\omega) = g(\Gamma, \omega)$. We note that, in [3], Amini–Caporaso defines the virtual weightless metric graph Γ^ω_ϵ, where each attached loop is assigned length $\epsilon > 0$. In this paper, we only use the case $\epsilon = 1$ (i.e. $\Gamma^\omega = \Gamma^\omega_1$).

Let e be a bridge of G. Let G_1, G_2 denote the connected components of $G \smallsetminus \{e\}$, which are equipped with the vertex-weight functions ω_1, ω_2 given by the restriction of ω. We say that e is a *positive-type* bridge if both (G_1, ω_1) and (G_2, ω_2) have genus at least 1.

We have a natural embeddings $\jmath : \Gamma \to \Gamma^\omega$ and $\jmath : \Gamma_{\mathbb{Q}} \to \Gamma^\omega_{\mathbb{Q}}$. Let $D \in \mathrm{Div}(\Gamma)$. Via $\jmath$, we have $\jmath_*(D) \in \mathrm{Div}(\Gamma^\omega)$. The rank $r_{(\Gamma,\omega)}(D)$ of D for (Γ, ω) is defined by

$$r_{(\Gamma,\omega)}(D) := r_{\Gamma^\omega}(\jmath_*(D)). \tag{2.1}$$

The canonical divisor K_{Γ^ω} has coefficient zero for any $v \in \Gamma^\omega \smallsetminus \Gamma$, so that K_{Γ^ω} is seen as a divisor on Γ. The Riemann–Roch formula on Γ^ω for $\jmath_*(D)$ gives rise to a formula

$$r_{(\Gamma,\omega)}(D) - r_{(\Gamma,\omega)}(K_{\Gamma^\omega} - D) = 1 - g(\Gamma, \omega) + \deg(D),$$

called the Riemann–Roch formula on (Γ, ω) for D.

Remark 2.11. Vertex-weighted graphs are generalization of finite graphs. Indeed, let G be a finite graph with associated metric graph Γ. Let $\mathbf{0} : V(G) \to \mathbb{Z}_{\geq 0}$ be the zero function. Then $(G, \mathbf{0})$ is a vertex-weighted graph, and we have $r_{(\Gamma,\mathbf{0})}(D) = r_\Gamma(D)$ for $D \in \mathrm{Div}(\Gamma)$. We will often identify a finite graph G with the vertex-weighted graph $(G, \mathbf{0})$ equipped with the zero function $\mathbf{0}$.

Vertex-weighted graphs naturally appear as reduction graphs of R-curves as we now explain. Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and residue field k as in §2.3. Let X be a geometrically irreducible smooth projective curve over $\mathbb{K}$, and $\mathscr{X}$ a semi-stable model of X over R. Let $\mathscr{X}_0$ is the special fiber of $\mathscr{X}$. Recall from §2.3 that we have the dual graph G of $\mathscr{X}_0$. Let v be a vertex of G, and let C_v be the corresponding irreducible component of $\mathscr{X}_0$. We define $\omega(v)$ to be the geometric genus of C_v. Then $\omega : V(G) \to \mathbb{Z}_{\geq 0}$ is a vertex-weight function, and we obtain a vertex-weighted graph (G, ω). We call (G, ω) the (vertex-weighted) *reduction graph* of $\mathscr{X}$.

Compared to G, the vertex-weighted graph (G, ω) captures more information of $\mathscr{X}$, encoding the genera of irreducible components of the special fiber. Amini–Caporaso [3] obtained Baker's specialization lemma for vertex-weighted graphs.

In the rest of this subsection, we show some properties of divisors on vertex-weighted graphs. Let (G, ω) be a vertex-weighted graph, with metric graph Γ associated to G. Let Γ^ω be the virtual weightless metric graph associated to (Γ, ω). Let $\jmath : \Gamma \to \Gamma^\omega$ be a natural embedding. Let $\jmath_* : \mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma^\omega)$ be the induced injective map.

Lemma 2.12. *We keep the notation above. Let $D \in \mathrm{Div}(\Gamma)$.*

1. *If $E \in \mathrm{Div}(\Gamma)$ satisfies $D \sim E$ on Γ, then $\jmath_*(D) \sim \jmath_*(E)$ on Γ^ω.*
2. *Fix a point $v_0 \in \Gamma$. Then D is a v_0-reduced divisor on Γ if and only if $\jmath_*(D)$ is a v_0-reduced divisor on Γ^ω.*
3. *$r_\Gamma(D) \geq 0$ if and only if $r_{(\Gamma,\omega)}(D) \geq 0$.*
4. *Let e be a leaf edge of G with leaf end v such that $\omega(v) = 0$. Let G_1 be the graph obtained by contracting e in G, Γ_1 the metric graph associated to G_1, and ω_1 the restriction of ω to $V(G_1)$. Let $\rho_1 : \Gamma \to \Gamma_1$ be the retraction map. Then $r_{(\Gamma,\omega)}(D) = r_{(\Gamma_1,\omega_1)}(\rho_{1*}(D))$.*

Proof. (1) Let f be a rational function on Γ such that $D - E = \mathrm{div}(f)$. For a loop $C \subset \Gamma^\omega$ that is added to a vertex $v \in \Gamma$ with positive weight, we set $\widetilde{f}(w) = f(v)$ for any $w \in C$. Then we obtain a rational function $\widetilde{f}$ on Γ^ω. Since $\jmath_*(D) - \jmath_*(E) = \mathrm{div}(\widetilde{f})$, we have $\jmath_*(D) \sim \jmath_*(E)$ on Γ^ω.

(2) Noting that Γ^ω is obtained by adding loops to Γ as one-point sums, the assertion follows from Lemma 2.9.

(3) The "only if" part is obvious. Indeed, if there exists an effective divisor D' on Γ with $D \sim D'$, then, by (1), $\jmath_*(D')$ is an effective divisor on Γ^w with $\jmath_*(D) \sim \jmath_*(D')$. Hence $r_{(\Gamma,\omega)}(D) = r_{\Gamma^\omega}(\jmath_*(D)) \geq 0$. We show the "if" part. Let v_0 be a point on Γ, and let E be the v_0-reduced divisor linearly equivalent to D on Γ. By (2), $\jmath_*(E)$ is a v_0-reduced divisor on Γ^ω, and by (1), $\jmath_*(E) \sim \jmath_*(D)$ on Γ^ω. Since $r_{(\Gamma,\omega)}(D) \geq 0$, Theorem 2.5 tells us that $\jmath_*(E)$ is effective. Hence E is also effective.

(4) The retraction map ρ_1 extends to the retraction map $\rho_1^\omega : \Gamma^\omega \to \Gamma_1^\omega$, where $e \subset \Gamma \subset \Gamma^\omega$ is retracted. Let $\jmath_1 : \Gamma_1 \hookrightarrow \Gamma_1^\omega$ be the natural embedding. Then Lemma 2.10 implies that

$$r_{(\Gamma,\omega)}(D) = r_{\Gamma^\omega}(\jmath_*(D)) = r_{\Gamma^\omega}\left(\rho_{1*}^\omega(\jmath_*(D))\right) = r_{\Gamma_1^{\omega_1}}\left(\jmath_{1*}(\rho_{1*}(D))\right) = r_{(\Gamma_1,\omega_1)}(\rho_{1*}(D)),$$

which completes the proof. $\square$

3. Hyperelliptic graphs

In this section, we collect some properties of hyperelliptic metric graphs. We define hyperelliptic vertex-weighted graphs and show its basic properties. We also define quantities p_Γ and $p_{(\Gamma,\omega)}$ for hyperelliptic graphs, which will play important roles in this paper.

3.1. **Hyperelliptic metric graphs.** We recall some properties of hyperelliptic metric graphs. We refer the reader to [8] and [12] for details.

We recall the definition of hyperelliptic metric graphs.

Definition 3.1 (Hyperelliptic metric graph, cf. [8, § 5.1] and [12, Definition 2.3])**.** A compact connected metric graph Γ is said to be *hyperelliptic* if if the genus of Γ is at least 2 and there exists a divisor on Γ with degree 2 and rank 1.

Definition 3.2 (Hyperelliptic finite graph, cf. [8, § 5.1] and [12, Definition 2.3])**.** Let G be a finite graph, and let Γ be the metric graph associated to G. A graph G is said to be *hyperelliptic* if Γ is hyperelliptic.

Assume that G is loopless. Originally, in [8], Baker–Norine define that G is hyperelliptic if there exists a divisor on D with degree 2 and rank 1. This is equivalent to the associated metric graph Γ of G being hyperelliptic. When G has a loop, this equivalence does not hold in general. In this paper, since we treat finite graphs with loops in general, we declare that a finite graph G is hyperelliptic by the above definition, which agrees with [8] for a loopless finite graph.

Let $\langle\iota\rangle$ be the group of order 2 with generator ι. We say that $\langle\iota\rangle$ acts non-trivially on Γ if there exists a group automorphism $\langle\iota\rangle \to \mathrm{Isom}(\Gamma)$, where $\mathrm{Isom}(\Gamma)$ is the group of isometries of Γ. Let $\Gamma/\langle\iota\rangle$ denotes the metric graph defined as the topological quotient with quotient metric. (Notice that our $\Gamma/\langle\iota\rangle$ is different from the one defined in [12, §2.2], which removes certain leaf edges from $\Gamma/\langle\iota\rangle$ to be compatible with the *loopless* quotient graph $G/\langle\iota\rangle$ defined in [8, §5.2].)

Definition 3.3 (Hyperelliptic involution)**.** Let Γ be a compact connected metric graph of genus at least 2. A *hyperelliptic* involution of Γ is an $\langle\iota\rangle$-action on Γ such that $\Gamma/\langle\iota\rangle$ is a tree.

First we study the action of involution on bridges.

Lemma 3.4. *Let Γ be a compact connected metric graph of genus at least 2 without points of valence 1. Assume that Γ has a hyperelliptic involution ι. Let e be an edge of Γ with endpoints v_1 and v_2. Then e is a bridge if and only if $\iota(e) = e$ and $\iota(v_i) = v_i$ for $i = 1, 2$.*

Proof. Since we assume that Γ has no points of valence 1, any bridge of Γ is positive type. Recall that we regard an edge e of Γ as a closed subset of Γ (so that e contains the endpoints of e), and let $\mathring{e}$ denotes the interior of e. (Recall also that by an edge of Γ, we mean an edge of the underlining finite graph of the canonical model of Γ.)

We first show the "if" part. Let e be an edge of Γ such that $\iota(e) = e$ and $\iota(v_i) = v_i$ for $i = 1, 2$. Since $\langle\iota\rangle$-action on $e\,(\subseteq \Gamma)$ is trivial and $\Gamma/\langle\iota\rangle$ is a tree, the metric graph $\Gamma \setminus \mathring{e}$ is not connected. Thus e is a bridge.

Next we show the "only if" part. Let e be a bridge with endpoints v_1 and v_2. Then one has $\Gamma \setminus \mathring{e} = \Gamma_1 \amalg \Gamma_2$ (disjoint union), where Γ_1 and Γ_2 are the connected components such that $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$.

Let us show that $\iota(e) = e$. To derive a contradiction, suppose that $\iota(e) \neq e$. Since $\iota(e) \neq e$, we may assume without loss of generality that $\iota(e) \subseteq \Gamma_2$. Then $e \cap \Gamma_1 = \{v_1\}$ and $\iota(e) \cap \Gamma_1 = \emptyset$. It follows that $e \cap \iota(\Gamma_1) = \emptyset$. Since $\iota(\Gamma_1)$ is connected and $e \cap \iota(\Gamma_1) = \emptyset$, we have either $\iota(\Gamma_1) \subseteq \Gamma_1$ or $\iota(\Gamma_1) \subseteq \Gamma_2$. The former does not occur, since $\iota(\Gamma_1) \subseteq \Gamma_1$ implies $\iota(\Gamma_1) = \Gamma_1$ (we apply ι), which contradicts $e \cap \Gamma_1 = \{v_1\} \neq \emptyset$. Thus we have $\iota(\Gamma_1) \subseteq \Gamma_2$, so that $\iota(\Gamma_1) \cap \Gamma_1 = \emptyset$. Since $\Gamma/\langle\iota\rangle$ is a tree, Γ_1 is a tree. Since Γ does not have points of valence 1, Γ_1 is not a point. Thus Γ_1 has at least 2 points of valence 1. On the other hand, Γ has no points of valence 1. Since Γ_1 is a connected component of $\Gamma \setminus \mathring{e}$, Γ_1 has at most one point of valence 1. This is a contradiction.

We conclude that $\iota(e) = e$. It remains to show that $\iota(v_1) = v_1$ and $\iota(v_2) = v_2$. It suffices to show $\iota(v_1) = v_1$, which amounts to $\iota(\Gamma_1) = \Gamma_1$. If $\iota(\Gamma_1) \neq \Gamma_1$, then the above argument implies that Γ_1 is a tree, which is a contradiction as before. This completes the the proof. □

The following theorem relates hyperelliptic metric graphs and hyperelliptic involutions.

Theorem 3.5 ([8, Theorem 5.12], [12, Theorem 3.13])**.** *Let Γ be a compact connected metric graph at least 2 without points of valence 1. Then the following are equivalent*:

(i) Γ *is hyperelliptic*;

(ii) Γ *has a hyperelliptic involution.*

Further, a hyperelliptic involution is unique.

Proof. By Lemma 3.4, we may assume that Γ is bridgeless. For the bridgeless case, see [8, Theorem 5.12] and [12, Theorem 3.13]. □

Remark 3.6. The uniqueness of hyperelliptic involution for hyperelliptic graphs is stated in [12, Corollary 3.9]. The proof there is based on [12, Proposition 3.8], and the proof of [12, Proposition 3.8] uses the Riemann–Roch formula on metric graphs. (The proof of idea is same as the proof of [8, Proposition 5.5].) However, it is not difficult to give a proof of the uniqueness without using the Riemann–Roch formula. Bellow, we give such proof to avoid arguing in a circle to apply Proposition 1.4.

Let us give a proof of the uniqueness of hyperelliptic involution without using the Riemann-Roch formula. To do that, we may assume that Γ is irreducible as we have Lemma 3.7. Note that the uniqueness is not used in the proof of Lemma 3.7.

Let Γ be an irreducible hyperelliptic graph of genus $g \geq 2$, and let ι and ι' be hyperelliptic involutions on Γ. Since $\Gamma/\langle\iota\rangle$ is a tree, one can show that there exists a point $v_0 \in \Gamma$ with $\iota(v_0) = v_0$ (see Lemma 3.14). Further, one has a marked special binary subgraph $(B, V(B), \iota|_B)$ of Γ with respect to ι (see § 5.2). Note that, in proving the existence of a special binary subgraph B, we only use the existence of a hyperelliptic involution, and we we do not need the uniqueness of a hyperelliptic involution.

We claim that $\iota'(B) = B$. Indeed, if $\iota'(B) \neq B$, we can see $\iota'(B) \cap B = \emptyset$. Then the image of B by the quotient map $\Gamma \to \Gamma/\langle\iota'\rangle$ is homeomorphic to B. Since $\Gamma/\langle\iota'\rangle$ is assumed to be a tree, this is a contradiction. Thus we have $\iota'(B) = B$.

We write $V(B) = \{v_B, v'_B\}$. Then we have $\iota(v_B) = v'_B$ by the definition of $(B, V(B), \iota|_B)$. We claim that $\iota'(v_B) = v'_B$. Indeed, if this is not the case, we have $\iota'(v_B) = v_B$. Since $\mathrm{val}(v_B) \geq 3$, v_B is a cut-vertex. This contradicts our assumption that Γ is irreducible.

Let $x \in \Gamma$ be an arbitrary point. Then we have $[\iota(v_B)] + [v_B] \sim [\iota(x)] + [x]$ and $[\iota'(v_B)] + [v_B] \sim [\iota'(x)] + [x]$. Note that those linear equivalence follows from the fact that $\Gamma/\langle\iota\rangle$ and $\Gamma/\langle\iota'\rangle$ are trees and that any two points in a tree are linearly equivalent to each other. Since $[\iota(v_B)] + [v_B] = [\iota'(v_B)] + [v_B]$, we obtain $[\iota(x)] + [x] \sim [\iota'(x)] + [x]$. Thus $[\iota(x)] \sim [\iota'(x)]$. Since Γ has no bridges, it follows that $\iota(x) = \iota'(x)$. Hence $\iota = \iota'$.

3.2. **Irreducible decomposition and contraction of hyperelliptic graphs.** The following lemmas show compatibility of the notion of being hyperelliptic under irreducible decomposition and contraction.

Lemma 3.7. *Let Γ be a hyperelliptic graph without points of valence 1. Let $\Gamma = \Gamma_0 \vee \cdots \vee \Gamma_n$ be the irreducible decomposition. Then Γ_i is a line segment, a circle or a hyperelliptic graph for any $i = 0, \ldots, n$. Moreover, if Γ_i is hyperelliptic, then the hyperelliptic involution on Γ induces the hyperelliptic involution on Γ_i.*

Proof. Any cut-vertex is ι-fixed by [12, Lemma 3.10] and Lemma 3.4. Thus the hyperelliptic involution ι preserves each Γ_i and the quotient $\Gamma_i/\langle\iota\rangle$ is a tree. It follows that, if $g(\Gamma_i) \geq 2$, then Γ_i with this ι-action is a hyperelliptic graph. This completes the proof. □

We see two properties of contraction on hyperelliptic metric graphs.

Lemma 3.8. *Let Γ be a compact connected metric graph. Suppose that Γ has a bridge, and let Γ_1 be the graph obtained by contracting a bridge. Then Γ is hyperelliptic if and only if Γ_1 is hyperelliptic.*

Proof. This follows from Lemma 2.10. □

Let Γ be a hyperelliptic metric graph. Let Γ' be the metric graph obtained by contracting all leaf edges of Γ. We denote by $\rho : \Gamma \to \Gamma'$ the retraction map, which induces $\rho_* : \mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma')$ on divisors. We note that Γ' is seen as a subgraph of Γ.

Lemma 3.9. *Let Γ' as above, and let $v, w \in \Gamma'$. Then $[v] + [\iota(v)] \sim [w] + [\iota(w)]$ as divisors on Γ'. Further, $[v] + [\iota(v)] \sim [w] + [\iota(w)]$ as divisors on Γ via the natural embedding $\Gamma' \hookrightarrow \Gamma$.*

Proof. Let $\overline{\Gamma}$ be the metric graph contracting all the bridges of Γ' and let $\overline{\rho}' : \Gamma' \to \overline{\Gamma}$ be the retraction map. Let $\overline{\iota}$ be the hyperelliptic involution of $\overline{\Gamma}$. By Lemma 3.4 and Theorem 3.5, we have $\overline{\rho}' \circ \iota' = \overline{\iota} \circ \overline{\rho}'$. Since $\overline{\rho}'(v) + \overline{\iota}(\overline{\rho}'(v)) \sim \overline{\rho}'(w) + \overline{\iota}(\overline{\rho}'(w))$ as a divisor on $\overline{\Gamma}$ by [12, Theorem 3.2 and its proof] (see also [8, Corollary 5.14]), we have $[v] + [\iota(v)] \sim [w] + [\iota(w)]$ as divisors on Γ' by Lemma 2.10. The second assertion follows from Lemma 2.10. □

3.3. **Hyperelliptic vertex-weighted graphs.** We define hyperelliptic vertex-weighted graphs. Since our focus on this paper is to prove Theorem 1.2, we restrict our attention to only necessary properties of hyperelliptic vertex-weighted graphs, which will be used later.

Definition 3.10 (Hyperelliptic vertex-weighted graph, cf. Definition 3.2)**.** Let (G, ω) be a vertex-weighted graph, and Γ the metric graph associated to G. We say that (G, ω) and (Γ, ω) are *hyperelliptic* if the genus of (G, ω) is at least 2 and there exists a divisor D on Γ such that $\deg(D) = 2$ and $r_{(\Gamma,\omega)}(D) = 1$.

Let (G, ω) be a vertex-weighted graph, and Γ the associated metric graph of G. Let Γ^ω be the virtual weightless metric graph associated to (Γ, ω). Recall that we have a natural embedding $\jmath : \Gamma \to \Gamma^\omega$ and they we denote by $\jmath_* : \mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma^\omega)$ the induced injective map.

Proposition 3.11. *Under the notation above, (Γ, ω) is hyperelliptic if and only if Γ^ω is hyperelliptic.*

Proof. The "only if" part is obvious. Indeed, suppose that (Γ, ω) is hyperelliptic, and we take a divisor D on Γ with $\deg(D) = 2$ and $r_{(\Gamma,\omega)}(D) = 1$. Since $r_{(\Gamma,\omega)}(D) = 1$ means by definition $r_{\Gamma^\omega}(\jmath_*(D)) = 1$, we see that $\jmath_*(D) \in \mathrm{Div}(\Gamma^\omega)$ is a divisor with $\deg \jmath_*(D) = 2$ and $r_{\Gamma^\omega}(\jmath_*(D)) = 1$. Thus Γ^ω is hyperelliptic.

We show the "if" part. Suppose Γ^ω is hyperelliptic. If ω is trivial, then there is nothing to prove, so that we assume that there exists a point $v_1 \in \Gamma$ with $\omega(v_1) > 0$. We put $D := 2[v_1]$. We are going to show that $r_{\Gamma^\omega}(D) = 1$.

Let $\overline{\Gamma^\omega}$ be the metric graph obtained from Γ^ω by contracting all the bridges, and let $\rho^\omega : \Gamma^\omega \to \overline{\Gamma^\omega}$ be the retraction map.

By Lemma 3.8 and Theorem 3.5, $\overline{\Gamma^\omega}$ is a hyperelliptic metric graph, and $\overline{\Gamma^\omega}$ has a (unique) hyperelliptic involution ι^ω. By Lemma 3.9, the divisor $D' := [\rho^\omega(v_1)] + [\iota^\omega(\rho^\omega(v_1))] \in \mathrm{Div}(\overline{\Gamma^\omega})$ has rank 1. Since we have add loops to the vertex v_1, v_1 is a cut-vertex of Γ^ω. Then $\rho(v_1)$ is a cut-vertex of $\overline{\Gamma^\omega}$. We then have $\iota^\omega(\rho^\omega(v_1)) = \rho^\omega(v_1)$ by [12, Lemma 3.9], so that $\rho^\omega_*(\jmath_*(D)) = 2[\rho^\omega(v_1)] = D'$. It follows that $r_{\overline{\Gamma^\omega}}(\rho_*(\jmath(D))) = 1$, and thus $r_{\Gamma^\omega}(D) = 1$ by Lemma 2.10. We obtain $r_{(\Gamma,\omega)}(D) = r_{\Gamma^\omega}(D) = 1$. □

The next proposition gives the vertex-weighted case of Theorem 3.5.

Proposition 3.12. *Let (G, ω) be a vertex-weighted graph of genus at least 2. Assume that any leaf end v of G satisfies $\omega(v) > 0$. Let Γ be the the associated metric graph of G, and Γ^ω the virtual weightless metric graph of (G, ω). Then the following are equivalent:*

(i) *(Γ, ω) is hyperelliptic;*
(ii) *Γ^ω has a unique hyperelliptic involution.*

Further, the hyperelliptic involution preserves Γ, where Γ is seen as a subgraph of Γ^ω via the natural embedding $\Gamma \hookrightarrow \Gamma^\omega$.

Proof. By the assumption of (G,ω), Γ^ω has no points of valence 1. Thus the condition (ii) is equivalent to Γ^ω being hyperelliptic, which, by Proposition 3.11, is equivalent to the condition (i).

Let ι^ω denote the hyperelliptic involution of Γ^ω. Recall that Γ^ω is obtained by adding loops to Γ. Let C be a loop which is added to a vertex $v \in V(G)$ with positive weight. To show that $\iota^\omega(\Gamma) = \Gamma$, it suffices to show that $\iota^\omega(C) = C$. Since v is a cut-vertex of Γ^ω and any cut-vertex is ι^ω-fixed by [12, Lemma 3.10], we have $\iota^\omega(v) = v$. Then $\iota^\omega(C)$ is a loop containing v. If $\iota^\omega(C) \neq C$, then $\Gamma^\omega/\langle\iota^\omega\rangle$ has a loop corresponding to C, which is impossible. Thus $\iota^\omega(C) = C$ and $\iota^\omega(\Gamma) = \Gamma$. □

Definition 3.13 (Hyperelliptic involution on a hyperelliptic vertex-weighted graph)**.** Let (G,ω) be a hyperelliptic vertex-weighted graph such that any leaf end v of G satisfies $\omega(v) > 0$, and Γ the metric graph associated to G. Let $\iota : \Gamma \to \Gamma$ be the involution defined by the restriction of the hyperelliptic involution of Γ^ω to Γ (cf. Proposition 3.12). We call ι the hyperelliptic involution of (Γ,ω).

We explain that the above definition is in accordance with Definition 3.3. Indeed, in the above definition, suppose that $g(\Gamma) \geq 2$. Since $\Gamma/\langle\iota\rangle$ is a subspace of a tree $\Gamma^\omega/\langle\iota^\omega\rangle$, $\Gamma/\langle\iota\rangle$ is also a tree. Thus ι is the hyperelliptic involution of Γ in the sense of Definition 3.3.

3.4. **Quantities $p_\Gamma(D)$ and $p_{(\Gamma,\omega)}(D)$.** We introduce a quantity $p_\Gamma(D)$ for a hyperelliptic metric graph Γ and a divisor D on Γ. We also introduce $p_{(\Gamma,\omega)}(D)$ for a hyperelliptic vertex-weighted graph (G,ω) and the corresponding metric graph Γ. These quantities will play important roles in this paper.

Let Γ be a hyperelliptic metric graph. Let Γ' be the metric graph obtained by contracting all leaf edges of Γ. We denote by $\rho : \Gamma \to \Gamma'$ the retraction map, which induces $\rho_* : \mathrm{Div}(\Gamma) \to \mathrm{Div}(\Gamma')$ on divisors. We note that Γ' is seen as a subgraph of Γ. Since Γ' is hyperelliptic, Γ' has a unique hyperelliptic involution ι' by Theorem 3.5.

We fix a point $v_0 \in \Gamma'$ with

$$\iota'(v_0) = v_0 \tag{3.2}$$

We note that such v_0 always exists (see Lemma 3.14 below). We regard v_0 as an element of Γ via the natural embedding $\Gamma' \hookrightarrow \Gamma$. For an effective divisor D on Γ, we set

$$p_\Gamma(D) = \max\{r \in \mathbb{Z}_{\geq 0} \mid |D - 2r[v_0]| \neq \emptyset\}. \tag{3.3}$$

We collect several results that will be used later.

Lemma 3.14. *Let Γ, Γ', ρ be as above.*

(1) *There always exist $v_0 \in \Gamma'$ with $\iota'(v_0) = v_0$.*
(2) *The quantity $p_\Gamma(D)$ defined in (3.3) is independent of the choice of $v_0 \in \Gamma'$ with $\iota'(v_0) = v_0$.*
(3) *For any effective divisor D on Γ, we have $p_\Gamma(D) = p_\Gamma(\rho_*(D))$.*

Proof. (1) Recall that $\langle\iota'\rangle$ acts non-trivially on Γ' and that $T' := \Gamma'/\langle\iota'\rangle$ is a tree. Let $\pi : \Gamma' \to T'$ be the quotient map. Take a leaf end $\pi(v_0) \in T'$. If $\pi^{-1}(\pi(v_0))$ consists of two points, then these two points should be leaf ends of Γ', but it contradicts the assuption on Γ'. Thus $\pi^{-1}(\pi(v_0)) = \{v_0\}$, which shows that $\iota'(v_0) = v_0$.

(2) Suppose $\widetilde{v_0} \in \Gamma'$ be another point with $\iota'(\widetilde{v_0}) = \widetilde{v_0}$. Then, by Lemma 3.9, we have $2[v_0] \sim 2[\widetilde{v_0}]$ in $\mathrm{Div}(\Gamma)$. Thus we obtain the assertion.

(3) We note that $D - 2r[v_0] \sim \rho_*(D) - 2r[v_0]$ in $\mathrm{Div}(\Gamma)$ by Lemma 2.10, from which the assertion follows. □

Now let (G,ω) be a hyperelliptic vertex-weighted graph, with corresponding metric graph Γ. Let G^ω be the virtual weightless graph, and Γ^ω the virtual weightless metric graph of (G,ω). By Proposition 3.11, Γ^ω is a hyperelliptic metric graph. Let $\jmath : \Gamma \hookrightarrow \Gamma^\omega$ be the natural embedding. For $D \in \mathrm{Div}(\Gamma)$, we set

$$p_{(\Gamma,\omega)}(D) := p_{\Gamma^\omega}(\jmath_*(D)). \tag{3.4}$$

4. Hyperelliptic semi-stable curves

In this section, we study hyperelliptic semi-stable curves, and equivariant deformation theory. Then we show Theorem 1.9 via equivariant deformation.

4.1. **Hyperelliptic semi-stable curves.** Let Ω be an algebraically closed field with $\mathrm{char}(\Omega) \neq 2$. Let $\mathcal{O}$ be a Ω-algebra. We call $\mathcal{O}$ a node if there is an isomorphism $\mathcal{O} \cong \Omega[[x,y]]/(xy)$ as an Ω-algebra. Let X_0 be an algebraic scheme of dimension 1 over Ω and let $c \in X_0$ be a closed point. We call c a node if the complete local ring $\widehat{\mathcal{O}_{X_0,c}}$ is a node in the above sense. A *semi-stable* curve is a connected, reduced, proper curve over Ω which has at most nodes as singular points. A *stable curve* over Ω is a semi-stable curve with ample dualizing sheaf. Recall that $\langle \iota \rangle$ denotes the group of order 2.

Definition 4.1 (Hyperelliptic curve)**.** A semi-stable (resp. stable) curve X_0 over Ω with an $\langle \iota \rangle$-action on X_0 is called a *hyperelliptic* semi-stable (resp. stable) curve if

(i) for any irreducible component C of X_0 with $\iota(C) = C$, the $\langle \iota \rangle$-action restricted to C is nontrivial (i.e., not the identity), and
(ii) $X_0/\langle \iota \rangle$ is a semi-stable curve of genus 0.

Example 4.2. Let X be a smooth connected projective curve of genus ≥ 2 define over Ω. Recall that X is *hyperelliptic* if there exists a divisor D on X such that $\deg(D) = 2$ and $r_{X_0}(D) = 1$. This definition is equivalent to the existence of an involution $\iota_X : X \to X$ for which the quotient X/ι_X is isomorphic to $\mathbb{P}^1$. The involution ι_X is unique and called the hyperelliptic involution of X. It follows that X, equipped with this $\langle \iota \rangle$-action, is a hyperelliptic stable curve in the sense of Definition 4.1.

Definition 4.3 (Hyperelliptic S-curve)**.** (1) Let $\mathscr{X} \to S$ be a proper and flat morphism over a scheme S. We say that $\mathscr{X}$ is a semi-stable S-curve (resp. a stable S-curve) if, for any point s in S, the geometric fiber of $\mathscr{X}_s$ is a semi-stable curve (resp. a stable curve).
(2) A semi-stable (resp. stable) S-curve $\mathscr{X}$ equipped with an $\langle \iota \rangle$-action on $\mathscr{X}/S$ is called a *hyperelliptic* semi-stable (resp. stable) S-curve if any geometric fiber of $\mathscr{X}_s$ equipped with the restriction of the $\langle \iota \rangle$-action is a hyperelliptic semi-stable curve.

As in the introduction, let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k. We assume that $\mathrm{char}(k) \neq 2$ in this subsection.

Proposition 4.4. *Let $\mathscr{X}$ be a semi-stable R-curve whose generic fiber is a smooth hyperelliptic curve X. Assume that there exists an $\langle \iota \rangle$-action on $\mathscr{X}/\mathrm{Spec}(R)$ such that the restriction of ι to the generic fiber is the hyperelliptic involution on X. Then $\mathscr{X}$ equipped with the $\langle \iota \rangle$-action is a hyperelliptic semi-stable R-curve.*

Proof. Let X_0 denote the the special fiber of $\mathscr{X} \to \mathrm{Spec}(R)$. Let C be an irreducible component of X_0 such that with $\iota(C) = C$. We show that the $\langle \iota \rangle$-action on C is nontrivial.

Let $q : \mathscr{X} \to \mathscr{Y}$ be the quotient by ι. Then, $q_* \mathcal{O}_{\mathscr{X}}$ is a coherent $\mathcal{O}_{\mathscr{Y}}$-module of rank 2. Let $\eta \in C$ be a general point in the regular locus of the special fiber. Then we have

$$\dim q^{-1}(q(\eta)) = \dim_{\kappa(q(\eta))} q_*(\mathcal{O}_{\mathscr{X}}) \otimes \kappa(q(\eta)) \geq 2,$$

where $\kappa(q(\eta))$ is the residue field at $q(\eta)$.

On the other hand, since $\mathrm{char}(k) \neq 2$, the order 2 of the action is invertible in R. Hence the restriction of q to the special fiber coincides with the quotient $X_0 \to X_0/\langle \iota \rangle$. Since $\eta \in C$ and $\dim q^{-1}(q(\eta)) \geq 2$, the $\langle \iota \rangle$-action on C cannot be trivial.

Next we show that $X_0/\langle \iota \rangle$ is a semi-stable curve of genus 0. Since the quotient of a semi-stable curve by a finite group is again semi-stable (see [22] for example), we obtain that $X_0/\langle \iota \rangle$ is a semi-stable curve.

Since $\mathscr{Y} \to \mathrm{Spec}(R)$ is flat (cf. [17, Proposition III.9.7]) and since the arithmetic genus of the generic fiber of $\mathscr{Y} \to \mathrm{Spec}(R)$ is 0, the arithmetic genus is of the special fiber $X_0/\langle\iota\rangle$ is also 0. We obtain that $X_0/\langle\iota\rangle$ is a semi-stable curve of genus 0. $\square$

4.2. **Equivariant specialization.** In this subsection, we prove Theorem 1.9. Let $\mathbb{K}, R, k$ be as in Theorem 1.2. In particular, we assume that $\mathrm{char}(k) \neq 2$.

Let (G, ω) be a vertex-weighted finite graph, and let Γ be the metric graph associated to G. We set

$$V(G_{\omega\circ}) := \{v \in V(G) \mid w(v) > 0 \text{ or } \mathrm{val}(v) \neq 2\}.$$

Let $(G_{\omega\circ}, \ell)$ be a length-weighted finite graph such that the set of vertices $V(G_{\omega\circ})$ of $G_{\omega\circ}$ is given above and that $(G_{\omega\circ}, \ell)$ is a model of Γ. We define the vertex-weight function $\omega : V(G_{\omega\circ}) \to \mathbb{Z}_{\geq 0}$ by the restriction to vertex-weight function $\omega : V(G) \to \mathbb{Z}_{\geq 0}$ to $V(G_{\omega\circ})$.

We call $(G_{\omega\circ}, \ell, \omega)$ the *vertex-weighted canonical model* of (Γ, ω), and call $(G_{\omega\circ}, \omega)$ the underlining vertex-weighted graph of the canonical model of (Γ, ω).

Lemma 4.5. *Let (G, ω) be a hyperelliptic vertex-weighted graph of genus g. Assume that any leaf end of v of G satisfies $\omega(v) > 0$. Let Γ be the metric graph associated to G, and $(G_{\omega\circ}, \omega)$ the underlining vertex-weighted graph of the canonical model of (Γ, ω). Then the following are equivalent.*

(1) *For any $v \in V(G_{\omega\circ})$, there are at most $(2\omega(v) + 2)$ positive-type bridges emanating from v.*
(2) *There exists a hyperelliptic stable curve X_0 of genus g such that*
   (i) *the (vertex-weighted) dual graph of X_0 is $(G_{\omega\circ}, \omega)$, and*
   (ii) *the $\langle\iota\rangle$-action on X_0 is compatible with the hyperelliptic involution on (Γ, ω) in the following sense: For any $v \in V(G_{\omega\circ})$, we have $\iota(C_v) = C_{\iota(v)}$, where C_v denotes the irreducible component of X_0 corresponding to v; For any $e \in E(G_{\omega\circ})$, we have $\iota(p_e) = p_{\iota(e)}$, where p_e is the node of X_0 corresponding to e.*

Proof. We show that (2) implies (1). We take $v \in V(G_{\omega\circ})$. If $\iota(v) \neq v$, then the proof of Proposition 3.12 shows $\omega(v) = 0$, so that there is no bridge emanating from v by our assumption. Thus it suffices to consider a vertex $v \in V(G_{\omega\circ})$ with $\iota(v) = v$. Let C_v be a corresponding irreducible component. Then ι acts on C_v as an involution and $C_v/\langle\iota\rangle$ is a smooth rational curve. By the Hurwitz formula, there exist exactly $(2\omega(v) + 2)$ ι-fixed smooth k-rational points on C_v. Since the node of X_0 on C_v corresponding to a bridge is an ι-fixed point, we cannot have more than $(2\omega(v) + 2)$ positive-type bridges. Hence we obtain (1).

Next we show that (1) implies (2). We consider a set $\{C_v\}_{v \in V(G_{\omega\circ})}$ of smooth projective curves over k indexed by $V(G_{\omega\circ})$ such that C_v has genus $\omega(v)$, and, furthermore, C_v is a hyperelliptic curve if $\omega(v) \geq 2$. For each $v \in V(G_{\omega\circ})$, let us fix an isomorphism $\iota_v : C_v \cong C_{\iota(v)}$ such that $\iota^2 = id$ and that $\iota_v \neq id$ if $\iota(v) = v$. For each pair $(v, e) \in V(G_{\omega\circ}) \times E(G_{\omega\circ})$, we fix a closed point $p_{v,e}$ of C_v such that the following conditions are satisfied:

- If $e \neq e'$ and v is an end-point of e and e', then $p_{v,e} \neq p_{v,e'}$.
- If v is the vertex on a loop edge e (in this case $\iota(v) = v$ and $\iota(e) = e$ automatically), then $\iota(p_{v,e}) \neq p_{v,e}$.
- If e is not a loop, then $\iota(p_{v,e}) = p_{\iota(v),\iota(e)}$.

Note that $p_{e,v}$ is ι-fixed if and only if e is not a loop, $\iota(e) = e$ and the two end-points of e are ι-fixed vertices. This is equivalent to saying that e is a bridge (cf. Lemma 3.4). Note also that, since there are at most $(2\omega(v) + 2)$ bridges emanating from v, we can take such $p_{v,e}$'s.

Now let X_0 be the curve obtained from the disjoint union of $\{C_v\}$ by identifying points by the equivalence relation $\sim$ given as follows:

- If e is a loop, then $p_{v,e} \sim \iota(p_{v,e})$.
- If e is not a loop, let v_1 and v_2 be the endpoints of e. Then $p_{v_1,e} \sim p_{v_2,e}$.

Then one can check that the vertex-weighted dual graph of X_0 is G_0. Further, we note that, when e is not a loop, we have chosen $p_{v,e}$ such that $\iota(p_{v,e}) = p_{\iota(v),\iota(e)}$. Thus we can put an $\langle\iota\rangle$-action that induces the $\langle\iota\rangle$-action on G_0. Hence we obtain (2). □

Theorem 4.6. *Let (G,ω) be a hyperelliptic vertex-weighted graph of genus $g(G,\omega) \geq 2$ such that every vertex v of G has at most $(2\omega(v)+2)$ positive-type bridges emanating from v. Assume that any leaf end v of G satisfies $\omega(v) > 0$. Let Γ be the metric graph associated to G. Then there exists a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G,w) such that the generic fiber X of $\mathscr{X}$ is hyperelliptic. Further, for the specialization map $\tau : X(\overline{\mathbb{K}}) \to \Gamma_{\mathbb{Q}}$, we have $\tau \circ \iota_X = \iota \circ \tau$, where ι_X is the hyperelliptic involution of X, and ι is the hyperelliptic involution of Γ.*

Proof. Let $(G_{\omega^\circ}, \ell, \omega)$ be the vertex-weighted canonical model of (Γ,ω). We take a hyperelliptic stable curve X_0 as in Lemma 4.5. Let $p_1,\ldots,p_r$ be the ι-fixed nodes of X_0 and let $p_{r+1},\ldots,p_s$ be the nodes such that $p_{r+1},\ldots,p_s,\iota(p_{r+1}),\ldots,\iota(p_s)$ are the distinct non-ι-fixed nodes.

Let π be a uniformizer of R. For $i=1,\ldots,r$, let $\widehat{\mathrm{Def}_{p_i,\iota}}(R)$ denote the profinite completion of the equivariant deformation $\widehat{\mathcal{O}_{X_0,p_i}}$ to R/π^n for $n \geq 1$. For $i = r+1,\ldots,r+s$, let $\widehat{\mathrm{Def}_{p_i}}(R)$ denote the profinite completion of the deformation $\widehat{\mathcal{O}_{X_0,p_i}}$ to R/π^n for $n \geq 1$ (see §A.2 for details). Let $\widehat{\Phi_\iota^{gl}} : \widehat{\mathrm{Def}_{(X_0,\rho_0)}} \to \prod_{i=1}^r \widehat{\mathrm{Def}_{(p_i,\iota)}} \times \prod_{i=r+1}^{r+s} \widehat{\mathrm{Def}_{p_i}}$ be the global-local morphism, which assigns to a deformation $\mathcal{X}$ of X_0 $\langle\iota\rangle$-equivariant deformation $\widehat{\mathcal{O}_{\mathcal{X},p_i}}$ of the node $\widehat{\mathcal{O}_{X_0,p_i}}$ for $1 \leq i \leq r$ and the deformation $\widehat{\mathcal{O}_{\mathcal{X},p_i}}$ of the node $\widehat{\mathcal{O}_{X_0,p_i}}$ for $r+1 \leq i \leq r+s$ (see §A.3 for details).

Let d_i' be an element in $\widehat{\mathrm{Def}_{p_i}}(R)$ that has a representative of form

$$\begin{array}{ccc} \widehat{\mathcal{O}_{X_0,p_i}} & \longleftarrow & R[[x,y]]/(xy-\pi^{l_i}) \\ \uparrow & & \uparrow \\ k & \longleftarrow & R, \end{array}$$

where l_i is the length of the edge of G_{ω° corresponding to a node p_i. Consider the case of $1 \leq i \leq r$. By Corollary A.7, there exists a lift $d_i \in \widehat{\mathrm{Def}_{p_i,\iota}}(R)$ of d_i'. In the case of $r+1 \leq i \leq r+s$, we put $d_i := d_i'$. Then we have the element $d = (d_i)$ in

$$\left(\prod_{i=1}^{r} \widehat{\mathrm{Def}_{(p_i,\iota)}} \times \prod_{i=r+1}^{r+s} \widehat{\mathrm{Def}_{p_i}}\right)(R).$$

By Corollary A.9, we can find an ι-equivariant diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spf}(R) \end{array}$$

whose isomorphism class in $\widehat{\mathrm{Def}_{(X_0,\iota)}}(R)$ is a lift of d by $\widehat{\Phi_\iota^{gl}}(R)$. This diagram of formal curves is algebraizable (cf. Remark A.3), and we write for the algebrization $\mathscr{X}' \to \mathrm{Spec}(R)$. Let $\mathscr{X} \to \mathrm{Spec}(R)$ be the minimal resolution of $\mathscr{X}' \to \mathrm{Spec}(R)$. Then the vertex-weighted reduction graph of it coincides with (Γ,ω). Thus $\mathscr{X} \to \mathrm{Spec}R$ is an R-curve with desired reduction graph.

It remains to show that the specialization map $\tau : X(\overline{\mathbb{K}}) \to \Gamma_{\mathbb{Q}}$ is compatible with the hyperelliptic involutions. To see that, let $\mathbb{K}'$ be a finite extension of $\mathbb{K}$ and R' be the ring of integer of $\mathbb{K}'$. Let $\mathscr{X}'' \to \mathrm{Spec}(R')$ be the base-change of $\mathscr{X}' \to \mathrm{Spec}(R)$ to $\mathrm{Spec}(R')$ and let $\widetilde{\mathscr{X}''}$ be the minimal resolution of $\mathscr{X}''$. Then the vertex-weighted dual graph of the special fiber $\mathscr{X}'' \to \mathrm{Spec}(R')$ is equal to $(G_{\omega^\circ},\omega)$. The vertex-weighted dual graph (G',ω') of the special fiber of $\widetilde{\mathscr{X}''} \to \mathrm{Spec}(R')$, where each edge is assigned length $1/[\mathbb{K}' : \mathbb{K}]$, is a model of (Γ,ω). The ι-action on $\mathscr{X}''$ lifts to that on $\widetilde{\mathscr{X}''}$. Let v' be a vertex of G' and let $C'_{v'}$ be the corresponding irreducible components in

the special fiber of $\widetilde{\mathscr{X}''} \to \mathrm{Spec}(R')$. Let e be an edge of G_{ω° with $v' \in e$ and p_e the corresponding node of X_0. From the construction of the hyperelliptic involution on X_0 in Lemma 4.5, we have $\iota_X(p_e) = p_{\iota(e)}$. Furthermore, we can check $\iota(C'_{v'}) = C'_{\iota(v')}$.

Let $P \in X(\overline{\mathbb{K}})$ be a point and take a finite extension $\mathbb{K}'$ so that $P \in X(\mathbb{K}')$. Then the corresponding section of $\widetilde{\mathscr{X}''} \to \mathrm{Spec}(R')$ intersects to a unique irreducible component $C'_{v'}$ for some $v' \in V(G')$. We have $\tau(P) = v'$ by definition. Since the section corresponding to $\iota(P)$ intersects with $\iota(C'_{v'})$ and since $\iota(C'_{v'}) = C'_{\iota(v')}$ as noted above, we obtain $\tau(\iota(P)) = \iota(v')$. □

We are ready to prove Theorem 1.9.

Corollary 4.7 (= Theorem 1.9)**.** *Let (G,ω) be a hyperelliptic vertex-weighted graph of genus $g(G,\omega) \geq 2$ such that every vertex v of G has at most $(2\omega(v)+2)$ positive-type bridges emanating from v. Then there exists a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G,ω) such that the generic fiber X of $\mathscr{X}$ is hyperelliptic.*

Proof. Successively contracting the leaf edges with a leaf end v of G such that $\omega(v) = 0$, we obtain a vertex-weighted hyperelliptic graph $(\overline{G}, \overline{\omega})$. Then we apply Theorem 4.6 to obtain a desired regular, semi-stable R-curve for $(\overline{G}, \overline{\omega})$. Taking successive blowing-ups, we obtain a desired R-curve for (G,ω). □

Remark 4.8. The above proofs of Theorem 4.6 and Theorem 1.9 use equivariant deformation. As we write in the Introduction, Amini–Baker–Brugallé–Rabinoff [2, Theorem 1.10] have shown Theorem 4.6 and Theorem 1.9 as a corollary of their deep studies of canonical gluing and star analytic spaces.

5. Reduced divisors on a hyperelliptic graph

In this section, we prove Theorem 1.10, which is a technical heart of our proof of Theorem 1.11 and of Theorem 1.2. We may assume that $D(v_0) = 0$ in proving Theorem 1.10, so we will show the following theorem.

Theorem 5.1. *Let Γ be a hyperelliptic metric graph of genus g. Let $v_0 \in \Gamma$ is a point as in* (3.2)*. Let $D \in \mathrm{Div}(\Gamma)$ be a v_0-reduced divisor on Γ such that $D(v_0) = 0$. Then, if $\deg(D) \leq g-1$, then there exists $w \in \Gamma \smallsetminus \{v_0\}$ such that $D + [w]$ is a v_0-reduced divisor.*

Remark 5.2. Under the notation in Theorem 5.1 (resp. Theorem 1.10), we always have $\deg(D) \leq g$ (resp. $\deg(D) \leq D(v_0) + g$). This follows from Theorem 1.11 or the Riemann–Roch formula on Γ (Theorem 2.3) .

We reduce Theorem 5.1 to a simpler case.

Lemma 5.3. *It suffices to prove Theorem 5.1 when Γ is an irreducible hyperelliptic graph without points of valence* 1.

Proof. **Step 1.** First we reduce ourselves to the case without points of valence 1. Let e be a leaf edge of Γ with leaf end v. Let Γ_1 be the metric graph that is obtained by contracting e, and $\rho_1 : \Gamma \to \Gamma_1$ be the retraction map. We regard Γ_1 as a subgraph of Γ via the natural emending $\Gamma_1 \hookrightarrow \Gamma$.

Suppose first that $v_0 \notin \mathring{e} \cup \{v\}$. By Lemmas 2.6 and 2.9, we have $D \in \mathrm{Div}(\Gamma_1)$ and D is v_0-reduced on Γ_1. Assuming Theorem 5.1 for Γ_1, we take $w \in \Gamma_1 \smallsetminus \{v_0\}$ such that $D+[w]$ is v_0-reduced on Γ_1. Then, by Lemma 2.9, $D+[w]$ is v_0-reduced on Γ.

Next suppose that $v_0 \notin \mathring{e} \cup \{v\}$. Let v_0' be the other endpoint of e different from v. By Lemma 2.6, we have $\mathrm{Supp}(D) \cap \left(\mathring{e} \cup \{v\}\right) \subseteq \{v_0\}$. Then we see that $\rho_{1*}(D)$ is v_0'-reduced. Assuming Theorem 5.1 for Γ_1, we take $w \in \Gamma_1 \smallsetminus \{v_0'\}$ such that $\rho_{1*}(D) + [w]$ is v_0'-reduced on Γ_1. Then, by Lemma 2.9, $\rho_{1*}(D) + [w]$ is v_0'-reduced on Γ. It follows that $D+[w]$ is v_0-reduced on Γ.

Thus we may assume that Γ has no points of valence 1 in proving Theorem 5.1.

Step 2. Next we reduce Theorem 5.1 to the irreducible case. Let Γ, v_0, D be as in Theorem 5.1. Let $\Gamma = \Gamma_0 \vee \cdots \vee \Gamma_n$ be the irreducible decomposition with $v_0 \in \Gamma_0$. Let $D_i = D|_{\Gamma_i}$ be the restriction of D to Γ_i. Then we have

$$\sum_{i=0}^{n} \deg(D_i) = \deg(D) + \sum_{i=1}^{n} D(v_i),$$

where the v_i are vertices in Lemma 2.9. We put $D'_0 := D_0$ and $D'_i := D_i - D(v_i)$ for $1 \leq i \leq n$. Then, for any $i \geq 0$, D'_i is an effective divisor on Γ_i with $D'_i(v_i) = 0$, and D'_i is a v_i-reduced divisor by Lemma 2.9. Since $\sum_{i=0}^{n} D'_i = \deg(D) \leq g-1$ and $g = g(\Gamma) = \sum_{i=0}^{n} g(\Gamma_i)$, there exists $j \in \{0, \ldots, n\}$ such that $\deg(D'_j) \leq g(\Gamma_j) - 1$. Since $g(\Gamma_j) > \deg(D'_j) \geq 0$, Γ_j is not a line segment, so that Lemma 3.7 tells us that Γ_j is an irreducible hyperelliptic graph or a circle.

We take a point w as follows: If Γ_j is a circle with a point $v_0 \in \Gamma_j$, then take any $w \in \Gamma_j \setminus \{v_j\}$. Assume that Γ_j is hyperelliptic and that Theorem 1.10 holds for Γ_j. Then there exists a point $w \in \Gamma_j \setminus \{v_j\}$ such that $D'_j + [w]$ is v_j-reduced, and we take such w. We set $E := D + [w]$. Then for $i \neq j$, $E|_{\Gamma_i} = D_i = D|_{\Gamma_i}$ is v_i-reduced by Lemma 2.9. For $i = j$, $E|_{\Gamma_j} = D_j + [w] = (D'_j + [w]) + D(v_j)[v_j]$ is v_j-reduced. By Lemma 2.9 again, we obtain that $E = D + [w]$ is v_0-reduced.

Thus Theorem 1.10 is reduced to the irreducible case. □

To prove Theorem 5.1 for an irreducible hyperelliptic graph without points of valence 1, we will use the induction on g, choosing a certain binary subgraph B of Γ, and comparing Γ with the metric graph obtained by contracting B. In the following, we first study non-saturated points on binary metric graphs, then we explain how we choose a certain binary subgraph B. Finally we give the proof of Theorem 5.1.

5.1. **Marked binary metric graph.** We study non-saturated points on binary metric graphs.

The *binary finite graph* G of genus $g \geq 1$, also called the *banana (finite) graph* of genus $g \geq 1$, is a finite graph with two vertices that are joined by $(g+1)$ edges.

A compact connected metric graph is a *binary metric graph* if there exists a homeomorphism f from the metric graph associated to the binary finite graph G. Let B be a binary metric graph of genus $g = g(B) \geq 1$. We set $V(B) = f(V(G))$, and we write $V(B) = \{v, v'\}$.

Suppose that $g(B) \geq 2$. Then $V(B)$ is the set of vertices of the underling binary finite graph of the canonical model of B. Further, B is a hyperelliptic graph and has a unique hyperelliptic involution ι_B. We have $\iota_B(v) = v'$.

Suppose $g(B) = 1$. Then B is homeomorphic to a circle. In this case, there exists a unique involution ι_B of B such that $B/\langle \iota_B \rangle$ is a tree and $\iota_B(v) = v'$.

Definition 5.4. We call the triple $(B, V(B), \iota_B)$ a *marked* binary metric graph. We note that if $g(B) \geq 2$, then $V(B)$ and ι are uniquely determined by B. On the other hand, if $g(B) = 1$, then $V(B)$ and ι depend on the choice of a homeomorphism f.

Lemma 5.5. *Let $(B, V(B), \iota_B)$ be a marked binary metric graph of genus $g(B) \geq 1$. We write $V(B) = \{v, v'\}$. Let D be an effective divisor on B. Assume that D satisfies the following conditions*:

(i) $D(v') = 0$*;*
(ii) *For any edge e of B,* $\deg(D|_e - D(v)[v]) \leq 1$*.*

Further, we assume that $\deg(D) = g(B)+1$*. Then there exists a unique compact subset $\emptyset \neq K \subsetneq B$ such that $v' \notin K$ and that every point of ∂K is saturated for D.*

Proof. To ease notation, we set $g = g(B)$. Let $e_1, \ldots, e_{g+1}$ be the edges of B. We first show the existence of K. Renumbering the edges if necessary, we write $D = D(v)[v] + \sum_{D(v)+1 \leq i \leq g+1} [v_i]$, where $v_i \in \mathring{e}_i$ for $D(v)+1 \leq i \leq g+1$. We put

$$K := \{v\} \cup \bigcup_{D(v)+1 \leq i \leq g+1} \overline{v_i v}, \tag{5.5}$$

where $\overline{v_i v}$ is the minimal path in e_i connecting v_i and v. We get

$$\partial K = \begin{cases} \{v, v_{D(v)+1}, \ldots, v_{g+1}\} & \text{if } D(v) > 0, \\ \{v_1, \ldots, v_{g+1}\} & \text{if } D(v) = 0. \end{cases}$$

We have $\mathrm{outdeg}_K^B(v_i) = 1 = D(v_i)$ for i with $D(v) + 1 \leq i \leq g+1$, and

$$\mathrm{outdeg}_K^B(v) = (g+1) - ((g+1) - (D(v))) = D(v).$$

Thus any point in ∂K is saturated for D. This proves the existence of K.

Next we show the uniqueness of K. Let $L \neq \emptyset$ be a compact subset such that $v' \notin L$ and that every point of ∂L is saturated for D. We will show that $L = K$. To see this, we may assume that L is connected.

We first show that $v \in L$. Indeed, to derive a contradiction, assume that $v \notin L$. Since $L \neq \emptyset$ and L is connected, there exists $i \in \{1, \ldots, g\}$ such that $L \subseteq e_i$. Since $v, v' \notin L$, L is a closed interval (possibly a point) contained in $\mathring{e}_i$. If L is not a point, then the coefficient of D at one of the endpoints of L is zero, so that this point is non-saturated for D. If $L = \{v\}$ is a point, then $\mathrm{outdeg}_L(v) = 2 > 1 \geq D(v)$, so that v is a non-saturated point for D. These contradict with the assumption of L. We conclude that $v \in L$.

For $i = 1, \ldots, g$, we set $L_i := L \cap e_i$. Since L is connected and $v' \notin L$, L_i is connected. Since $v' \notin L$ and $v \in L$, there exists $w_i \in e_i \setminus \{v'\}$ such that $L_i = \overline{v w_i}$.

Suppose that $1 \leq i \leq D(v)$. In this case, if $w_i \neq v$, then $D(w_i) = 0$. Since $w_i \in \partial L$ is saturated for D, we must have $w_i = v$, i.e., $L_i = \{v\}$.

Suppose next that $D(v) + 1 \leq i \leq g+1$. Since $L_1 = \cdots = L_{D(v)} = \{v\}$, we have $\mathrm{outdeg}_L^B(v) \geq D(v)$. Since v is saturated for D, we get $\mathrm{outdeg}_L^B(v) = D(v)$. Then $w_i \neq v$ for any i with $D(v)+1 \leq i \leq g+1$. Since $w_i \in \partial L$ and is saturated for D, we obtain $w_i = v_i$. In conclusion, L is of the form of the right hand side of (5.5). This completes the proof of the uniqueness of K. □

Remark 5.6. In Lemma 5.5, we see from (5.5) that $\mathrm{outdeg}_K^B(v) = D(v)$.

Corollary 5.7. *Let B, $V(B) = \{v, v'\}$ and ι_B be those as in Lemma 5.5. Let D be an effective divisor on B. Assume that D satisfies the conditions* (i)(ii) *in Lemma 5.5. Further, we assume that* $\deg(D) \leq g(B)$. *Let $\emptyset \neq A \subseteq B$ be a compact subset with $v' \notin A$. Then ∂A has a non-saturated point for D.*

Proof. To ease notation, we set $g = g(B)$. We put $E := D + (g+1-\deg D)[v]$. Then we have $E(v) \neq 0$ and $\deg(E) = g+1$. Note that $E(v') = 0$ and E also satisfies the conditions (i)(ii) of Lemma 5.5. By Lemma 5.5, there exists a unique non-empty compact subset $K \subsetneq B$ such that $v' \notin K$ and any point in ∂K is saturated for E. If $A \neq K$, then the uniqueness of K implies that ∂A has a non-saturated point for E, hence for D. If $A = K$, then we have $\mathrm{outdeg}_K^B(v) = E(v)$ (cf. Remark 5.6). Then $\mathrm{outdeg}_K^B(v) > D(v)$, and $v \in \partial A$ is a non-saturated point for D. □

5.2. **Special binary subgraph.** In the proof of Theorem 1.10, we focus on a subgraph of Γ, which we call a special binary subgraph. Here we give its definition and study some properties.

Throughout this subsection, we assume that Γ is an irreducible hyperelliptic graph without points of valence 1. By Theorem 3.5, Γ has a unique hyperelliptic involution ι. We fix a point $v_0 \in \Gamma$ with $\iota(v_0) = v_0$.

Let B be a compact connected subspace of Γ. We say that B is a v_0-*special binary subgraph* if the following three conditions are satisfied:

(i) $(B, \partial B, \iota|_B)$ is a marked binary metric graph with $v_0 \notin B$.
(ii) B is maximal in the sense that if $B \subseteq B'$ and $(B', \partial B', \iota|_{B'})$ is a marked binary metric graph with $v_0 \notin B'$, then $B = B'$.
(iii) We set $\partial B = \{v_B, \iota_B(v_B)\}$. Then $\mathrm{outdeg}_B^\Gamma(v_B) = \mathrm{outdeg}_B^\Gamma(\iota_B(v_B)) = 1$.

Lemma 5.8. *There exists a v_0-special binary subgraph.*

Proof. Let $(G_\circ, \ell)$ be the canonical model of Γ. Since Γ is assumed to be irreducible (i.e. without cut-vertices), $G_\circ$ is loopless. Let $\pi : \Gamma \to T := \Gamma/\langle\iota\rangle$ be the quotient map.

Let $v \in V(G)$. Since Γ is assumed to have no points of valence 1, we have $\mathrm{val}(v) \geq 3$.

We first show that $\iota(v) \neq v$. Indeed, suppose that $\iota(v) = v$. Let $e_1, \cdots, e_{\mathrm{val}(v)}$ be the edges of $G_\circ$ with endpoint v. Since $\mathrm{val}(v) \geq 3$, $\pi(e_1), \cdots, \pi(e_{\mathrm{val}(v)})$ are not the same. Since T is a tree, $T \smallsetminus \{\pi(v)\}$ is then disconnected. Hence v is a cut-vertex of Γ, which contradicts with the irreducibility of Γ.

Let e be an edge of $G_\circ$ with endpoints v_1, v_2. We say that e is an *m-edge* (here "m" stands for "movable") if $\iota(v_1) \neq v_2$. We say that e is an *f-edge* (here "f" stands for "fixed") if $\iota(v_1) = v_2$. If e is an m-edge, then $\iota(e) \cap e = \emptyset$. If e is an f-edge, then $\iota(e) = e$.

We claim that there exists a vertex v of G with at most one m-edge emanating from v. Indeed, if v has more than one m-edges, then we take an m-edge m_1 such that $\pi(m_1) \smallsetminus \{\pi(v)\}$ lies a connected component of $T \smallsetminus \{\pi(v)\}$ which does not contain v_0. Then we replace v with the other endpoint of m_1. This procedure stops eventually, and we obtain a vertex v with at most one m-edge emanating from v.

We fix such v. Let $\ell_1, \ldots, \ell_s$ be the f-edges emanating from v such that $\ell_i \not\ni v_0$ for any i. Since $\mathrm{val}(v) \geq 3$, we have $s \geq 2$. Then $B := \ell_1 \cup \cdots \cup \ell_s$ is a binary metric graph with marking $V(B) := \{v, \iota(v)\}$ and $\iota_B := \iota|_B$. From the construction, B is a v_0-special binary subgraph. □

Remark 5.9. Suppose that Γ is not a binary metric graph. Let e_0 be the vertex with $v_0 \in e_0$. Then we can take a vertex v such that v is not an endpoint of e_0 and $\iota(v) \neq v$. Then the last two paragraphs of the proof of Lemma 5.8 show that there exists a v_0-special binary subgraph B such that $B \cap e_0 = \emptyset$. Note that there exists an m-edge emanating from a vertex of the marked binary metric graph B. *In the sequel, when the hyperelliptic graph Γ is not a binary metric graph, we always take such a v_0-special binary subgraph B.*

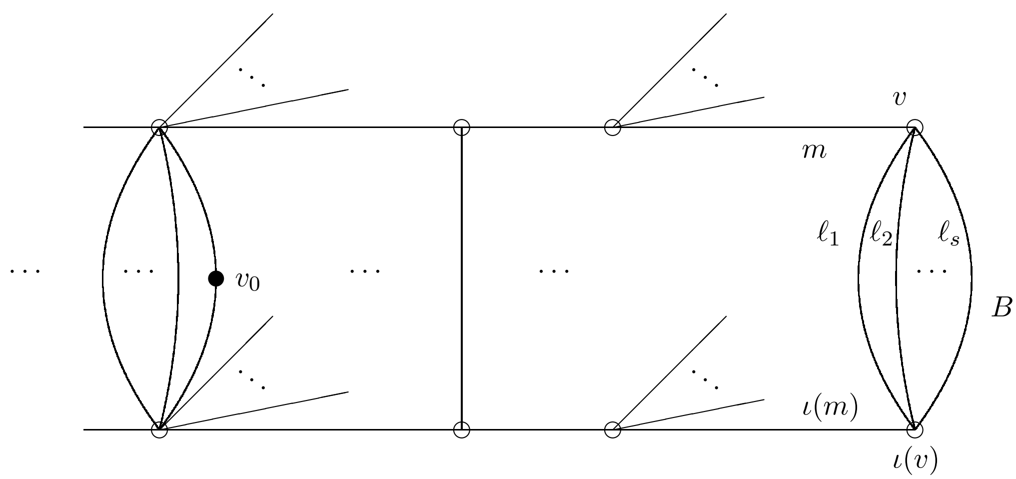


FIGURE 1. An irreducible hyperelliptic graph Γ without points of valence 1. The hyperelliptic involution ι is given by the reflection relative to the horizontal line through v_0. Horizontal edges are m-edges, while vertical edges are f-edges. The binary metric graph on the right is a v_0-special binary subgraph B.

Lemma 5.10. *Let B be a v_0-special binary subgraph, and we set $\iota_B = \iota|_B$ and $V(B) := \partial B = \{v_B, \iota_B(v_B)\}$. Let D be a v_0-reduced divisor on Γ. Then the divisor $D|_B$, which is the restriction of D to B, satisfies the following two conditions*:

(i) $V(B) \not\subseteq \mathrm{Supp}(D|_B)$.

(ii) *For any edge e of B,* $\deg\left(D|_e - D(v_B)[v_B] - D\left(\iota_B(v_B)\right)[\iota_B(v_B)]\right) \leq 1$.
Further, we have $\deg(D|_B) \leq g(B)+1$.

Proof. If $D(v_B) > 0$ and $D(\iota(v_B)) > 0$, then ∂B does not have a non-saturated point for D. Since B is a compact subset of $\Gamma \smallsetminus \{v_0\}$, this contradicts with the assumption that D is v_0-reduced. Thus we get (i). Suppose that there exist an edge e of B and two points $w_1, w_2 \in \mathring{e}$ with $D(w_1), D(w_2) > 0$. Let A be the closed path $\overline{w_1 w_2}$ connecting w_1 and w_2. Then $\partial A = \{w_1, w_2\}$ and w_1, w_2 are saturated for D. Since $v_0 \notin A$, this is a contradiction. Thus we get (ii).

To obtain the second assertion, we may assume that $D(\iota(v_B)) = 0$ by the condition (i). We set $E := D|_B + (g(B)+1-\deg(D|_B))[v_B]$. Since $D|_B$ satisfies the condition (i)(ii), we see that E is effective and it satisfies the condition (i)(ii) in Lemma 5.5. Let K be a compact set of B in Lemma 5.5 for E. Since $\mathrm{outdeg}_B^{\Gamma}(v_B) = 1$, we see that $\deg(D|_B) > g(B)+1$ implies any point in ∂K is saturated for D. Since D is v_0-reduced, this is a contradiction. Thus $\deg(D|_B) \leq g(B)+1$. □

In the rest of this section, we fix the following notation. We define the metric graph $\Gamma_{/B}$ as the metric space obtained from Γ by contracting B to a point. Let

$$\varphi : \Gamma \to \Gamma_{/B}$$

be the canonical continuous map. We set $\{v'_\infty\} = \varphi(B)$. From the condition (iii) of the v_0-special binary subgraph, we see that v'_∞ has valence 2.

Lemma 5.11. *Assume that Γ is not a binary metric graph. Let B be such a v_0-special binary subgraph on Γ as noted in Remark 5.9. Then $g(\Gamma_{/B}) = g(\Gamma) - g(B) \geq 2$. Further, $\Gamma_{/B}$ is an irreducible hyperelliptic metric graph without points of valence* 1.

Proof. The equality $g(\Gamma_{/B}) = g(\Gamma) - g(B)$ is obvious from the construction. We take a point $u_0 \in B$ and consider a u_0-special binary subgraph B'. Since Γ is assumed not to be a binary metric graph, we see that $B \cap B' = \emptyset$. Thus $g(\Gamma_{/B}) \geq 2$.

Since the ι-action preserves B, and since $B/\langle\iota\rangle$ is a union of leaf edges of $\Gamma/\langle\iota\rangle$, there exists an ι-action on $\Gamma_{/B}$ such that $\Gamma_{/B}/\langle\iota\rangle$ is a tree. Since Γ has no points of valence 1, so does $\Gamma_{/B}$. It follows from Theorem 3.5 that $\Gamma_{/B}$ is hyperelliptic. Since Γ is irreducible and without points of valence 1, so is $\Gamma_{/B}$. □

For a divisor $D \in \mathrm{Div}(\Gamma)$, we set

$$D' = \sum_{v \in \Gamma \smallsetminus B} D\left(\varphi(v)\right)[\varphi(v)] + (\deg(D|_B) - g(B))\,[v'_\infty] \in \mathrm{Div}(\Gamma_{/B}). \tag{5.6}$$

We note that if $\deg D|_B = g(B)$, then $D'(v'_\infty) = 0$, and if $\deg D|_B = g(B)+1$, then $D'(v'_\infty) = 1$. Also, if $\deg(D) \leq g(\Gamma) - 1$, then we have

$$\deg(D') = \deg(D) - g(B) \leq (g(\Gamma)-1) - g(B) = g(\Gamma_{/B}) - 1. \tag{5.7}$$

Lemma 5.12. *Let B be a v_0-special binary subgraph on Γ. Let D be a divisor on Γ satisfying the conditions* (i)(ii) *in Lemma 5.10.*

(1) *Suppose that* $\deg(D|_B) = g(B)$. *Then, if D is v_0-reduced, then D' is $\varphi(v_0)$-reduced.*
(2) *Suppose that* $\deg(D|_B) = g(B)+1$. *Then D is v_0-reduced if and only if D' is $\varphi(v_0)$-reduced.*

Proof. (1) Suppose that D is v_0-reduced. Let $A' \subseteq \Gamma'$ be a non-empty compact subset with $A' \smallsetminus \{\varphi(v_0)\}$. We will show that $\partial A'$ has a non-saturated point for D'.

Suppose first that $v'_\infty \notin \partial A'$. We put $A := \varphi^{-1}(A')$, and then $(\partial A) \cap B = \emptyset$. Since D is v_0-reduced, there exists a non-saturated point $a \in \partial A$ for D. Then $a' := \varphi(a)$ is in $\partial A'$ and is non-saturated for D'. Suppose next that $v'_\infty \in \partial A'$. Since $D'(v'_\infty) = 0$, $v'_\infty \in \partial A'$ is non-saturated for D'. This completes the proof of (1).

(2) We show the "only if" part. As in (1), let D be v_0-reduced, and let $A' \subseteq \Gamma'$ be a non-empty compact subset with $A' \smallsetminus \{\varphi(v_0)\}$. We will show that $\partial A'$ has a non-saturated point for D'. The argument in (1) show that it suffices to treat when $v'_\infty \in \partial A'$.

Since $v'_\infty \in \partial A'$, there exists a unique vertex v_B on B such that $\varphi^{-1}(A' \setminus \{v'_\infty\}) \cup \{v_B\}$ is the closure of $\varphi^{-1}(A' \setminus \{v'_\infty\})$ in Γ. We define $A \subseteq \Gamma$ as follows:

- If $D(\iota(v_B)) \neq 0$, then we put $A := \varphi^{-1}(A')$.
- If $D(\iota(v_B)) = 0$, then let K be the non-empty compact subset with $\iota(v_B) \notin K$ in Lemma 5.5 for $D|_B$. We put $A := (\varphi^{-1}(A') \setminus B) \cup K$.

We claim that all the points in $(\partial A) \cap B$ are saturated for D. To see this, suppose first that $D(\iota(v_B)) \neq 0$. Then, by the definition of A, we see that v_B is an interior point of A and we have $(\partial A) \cap B = \{\iota(v_B)\}$. It follows that $\mathrm{outdeg}_A(\iota(v_B)) = 1 \leq D(\iota(v_B))$, so that the unique point of $(\partial A) \cap B$ is saturated for D.

Suppose next that $D(\iota(v_B)) = 0$. Then, by the explicit description (5.5) of K, we get $(\partial A) \cap B = \partial^B K$, where $\partial^B K$ is the boundary of K in B. Let $a \in (\partial A) \cap B$. If $a = v_B$, then $\mathrm{outdeg}^\Gamma_A(v_B) = \mathrm{outdeg}^B_{A\cap B}(v_B) = D(v_B)$ (cf. Remark 5.6), so that $a = v_B$ is saturated for D. Since $\iota(v_B) \notin K$, we have $a \neq \iota(v_B)$. Finally, if $a \neq v_B, \iota(v_B)$, then $\mathrm{outdeg}_A(a) = D(a)$ and a is saturated for D. Hence we obtain the claim.

Since D is assumed to be v_0-reduced, there exists a non-saturated point $a \in (\partial A) \setminus B$ for D. Then $\varphi(a) \in \partial A'$ is a non-saturated point for D'. It follows that D' is v'_0-reduced.

We will show the "if" part, so that we assume that D' is v'_0-reduced. Let $A \subseteq \Gamma \setminus \{v_0\}$ be a non-empty compact connected subset. We put $A' := \varphi(A)$. Since D' is v'_0-reduced, there exists a non-saturated point $a' \in \partial A'$ for D'.

Case 1. Suppose that $v'_\infty \notin \partial A'$. In this case, we have $A \setminus B \approx A' \setminus \{v'_\infty\}$ and $a' \neq v'_\infty$. We set $a = \varphi^{-1}(a')$. Then $a \in \partial\varphi^{-1}(A')$ is non-saturated for D. Since $\partial A' \cap \{v'_\infty\} = \emptyset$, we get $\partial\varphi^{-1}(A') \subseteq \partial A$. Then $a \in \partial A$, and a is non-saturated for D. Thus D is v_0-reduced in this case.

Case 2. Suppose that $\{v'_\infty\} \subsetneq \partial A'$. In this case, v'_∞ belongs to A' but is not isolated in A'. Since v'_∞ has valence 2, we get $\mathrm{outdeg}_{A'}(v'_\infty) = 1$. Since $D'(v'_\infty) = 1$, it follows that v'_∞ is saturated for D'. Since a' is taken as a saturated point for D', we have $a' \neq v'_\infty$. We set $a = \varphi^{-1}(a')$. Then $a \in \partial A$, and a is non-saturated for D. Thus D is v_0-reduced in this case, too.

Case 3. Suppose that $\{v'_\infty\} = \partial A'$. In this case, $A \subseteq B$. We write $\partial B = \{v_B, \iota(v_B)\}$ as before. Since D satisfies the condition (i) in Lemma 5.10, interchanging v_B and $\iota(v_B)$ if necessary, we may assume that $D(\iota(v_B)) = 0$. If $\iota(v_B) \in A$, then it follows from $A \subseteq B$ and $v_B \in \partial B$ that $\iota(v_B) \in \partial A$. Since $\mathrm{outdeg}^\Gamma_A(\iota(v_B)) \geq 1 > 0 = D(\iota(v_B))$, we see that $\iota(v_B)$ is non-saturated for D. Suppose that $\iota(v_B) \notin A$. If $\partial^B A$ has a non-saturated point a for D, then $a \in \partial A$ and we are done. Therefore we may and do assume that any point of $\partial^B A$ is saturated for D. Then Lemma 5.5 gives that A coincides with the unique K in Lemma 5.5, we have $\mathrm{outdeg}^B_A(v_B) = D(v_B)$ (cf. Remark 5.6). Since $\mathrm{outdeg}_A(v_B) = \mathrm{outdeg}^B_A(v_B) + 1$, we see that v_B is non-saturated for D. This completes the proof. $\square$

5.3. **Proof of Theorem 5.1.** Let Γ be a hyperelliptic metric graph of genus $g(\Gamma) \geq 2$. We prove Theorem 5.1 on $g(\Gamma)$.

By Lemma 5.3, we may and do assume that Γ *is irreducible and without vertices of points* 1. Let ι be the hyperelliptic involution of Γ, and let v_0 be a point of Γ with $\iota(v_0) = v_0$ (cf. Theorem 3.5). Let $D \in \mathrm{Div}(\Gamma)$ be a v_0-reduced divisor with $D(v_0) = 0$ and $\deg(D) \leq g(\Gamma) - 1$.

We fix a v_0-special binary subgraph B of Γ. We recall some notation from the previous subsection. The metric graph $\Gamma_{/B}$ is obtained from Γ by contracting B to a point, and $\varphi : \Gamma \to \Gamma_{/B}$ is the quotient map, and $\{v'_\infty\} := \varphi(B)$. Also, e'_∞ is the unique edge of the canonical model of $\Gamma_{/B}$ that contains $\{v'_\infty\}$ as an interior point. Let $D' \in \mathrm{Div}(\Gamma_{/B})$ be the divisor defined by (5.6). We will compare D and D'.

Note that, it Γ is not a binary metric graph, then $\Gamma_{/B}$ is again an irreducible hyperelliptic metric graph without points of valence 1 by Lemma 5.11. Thus our induction step starts when Γ is a binary metric graph of genus $g \geq 2$.

For a subset $C \subseteq B$, we denote by $\partial^B C$ the set of boundary points of C in B.

Lemma 5.13. *Theorem 5.1 holds true when Γ is a binary metric graph of genus $g(\Gamma) \geq 2$.*

Proof. Since $g(\Gamma) \geq 2$, the binary metric graph Γ has natural marking $V(\Gamma) = \{v, v'\}$ and hyperelliptic involution ι. Let e be an edge of Γ with $v_0 \in \Gamma$. We set $B = \Gamma \setminus \mathring{e}$. Then $(B, \{v, v'\}, \iota|_B)$ is a marked binary metric graph. Note that B is a v_0-special binary subgraph of Γ.

Since $\deg(D|_B) \leq \deg(D) \leq g(\Gamma) - 1 = g(B)$, we see that $D|_B$ satisfies (i)(ii) in Lemma 5.10. Since D is v_0-reduced and $D(v_0) = 0$, we see that $0 \leq \deg(D|_e - D(v)[v] - D(v')[v']) \leq 1$. Therefore we find that D satisfies the conditions (i)(ii) in Lemma 5.5. Further, we have $\mathrm{Supp}(D) \cap \overline{v_0 v} = \emptyset$ or $\mathrm{Supp}(D) \cap \overline{v_0 v'} = \emptyset$. We may assume that $\mathrm{Supp}(D) \cap \overline{v_0 v'} = \emptyset$ without loss of generality.

Let us show that $D + [v]$ is v_0-reduced. To do that, let $A \subset \Gamma \setminus \{v_0\}$ be a connected compact subset, and we show that ∂A has a point non-saturated for $D + [v_0]$.

Case 1. Suppose that $v' \in A$. Since A is connected, we can write $A \cap e = \overline{wv'}$ for some $w \in e$ (possibly $w = v'$). Then $w \in \partial A$, and w is non-saturated for $D + [v]$ since $(D + [v])(w) = 0$.

Case 2. Suppose that $v' \notin A$. We put $E := D + (g(\Gamma) + 1 - \deg(D))[v]$. We have $\deg(E) = g(\Gamma) + 1 - \deg(D)$. Since $\deg(D) \leq g(\Gamma) - 1$, E is effective. We take the compact subset $K \subset \Gamma$ as in Lemma 5.5 for E. This K is characterized by the condition that $v' \in K$ and that any point of ∂K is saturated for E. Suppose $A \neq K$. Then ∂A has a point non-saturated point a for E by the uniqueness of K. Since $D + [v] \leq E$, a is also non-saturated for $D + [v]$, and thus we are done. Suppose $A = K$. Since $E(v) > 0$, we see $v \in \partial K$ from the explicit expression in (5.5). We claim v is saturated for $D + [v]$. Indeed, we have $(D + [v])(v) = D(v) + 1$ on one hand, and $\mathrm{outdeg}_K^\Gamma(v) = D(v) + g(\Gamma) + 1 - \deg(D)$ by Remark 5.6 on the other hand. Since $g(\Gamma) > \deg(D)$ by our assumption, the two equality above give us $\mathrm{outdeg}_K^\Gamma(v) > (D + [v])(v)$. Thus v is a point in ∂A non-saturated for $D + [v]$. $\square$

We now prove Theorem 5.1 by induction. We may and do assume that Γ is not a binary metric graph by Lemma 5.13. As noted in Remark 5.9, we can take a v_0-special binary subgraph B of Γ in such a way that there is an m-edges emanating from a vertex of B. From here on, we assume that B is such a v_0-special binary subgraph.

Case 1. Suppose that $\deg(D|_B) \leq g(B) - 1$. Since D is v_0-reduced, $D|_B$ satisfies the conditions (i)(ii) of Lemma 5.10. Since $\deg(D|_B) \leq g(B) - 1$, there exists an edge e_1 such that $\mathring{e}_1 \cap \mathrm{Supp}\, D = \emptyset$. Let w be the midpoint of e_1. We put $E := D + [w]$. We claim that E is v_0-reduced. Indeed, let A be a compact connected subset of $\Gamma \setminus \{v_0\}$. We will show that ∂A has a non-saturated point for E. Since D is assumed to be v_0-reduced, it suffices to consider the case of $w \in \partial A$.

Suppose first that $\partial B \not\subseteq A$. Since E satisfies the conditions (i)(ii) of Lemma 5.10 and $\deg(E)|_B \leq g(B)$, Corollary 5.7 tells us that there exists a point $a \in \partial^B(A \cap B)$ that is non-saturated for E. Suppose next that $\partial B \subseteq A$. We put $A_1 := A \cup B$. Since D is v_0-reduced, there exists $a \in \partial A_1$ that is a non-saturated point for D. Since w is an interior point of B, we have $w \notin \partial A_1$, so that a is a non-saturated point for E. Since $\partial B \subset A$, we have $\partial A_1 \subset \partial A$ and hence $a \in \partial A$. Thus ∂A has a non-saturated point a for E. This completes the proof of Theorem 5.1 for Case 1.

The case $\deg(D|_B) = g(B)$ will be the most complicated case, so we next consider the case $\deg(D|_B) \geq g(B) + 1$.

Case 2. Suppose that $\deg(D|_B) \geq g(B) + 1$. By Lemma 5.10, we have $\deg(D|_B) = g(B) + 1$. By Lemma 5.12(1), D' is a v_0'-reduced divisor. Since $\deg D' \leq g(\Gamma_{/B}) - 1$ by (5.7), by induction, there exists $w' \in \Gamma_{/B}$ such that $D' + [w']$ is a v_0'-reduced divisor on $\Gamma_{/B}$. We set $w = \varphi^{-1}(w') \in \Gamma$. Then Lemma 5.12(2) shows that $D + [w]$ is a v_0-reduced divisor on Γ. This complete the proof of Theorem 5.1 for Case 2.

Case 3. Suppose that $\deg(D|_B) = g(B)$. If there exists a point $w \in B$ such that $D + [w]$ is v_0-reduced, then we are done. In what follows, we make the following assumption.

Assumption 5.13.1. $D + [v] \in \mathrm{Div}(\Gamma)$ is not v_0-reduced for any $v \in B$.

The divisor $D' \in \mathrm{Div}(\Gamma_{/B})$ defined in (5.6) satisfies $\deg(D') \leq g(\Gamma_{/B})-1$ by (5.7) and $D'(\varphi(v_0)) = 0$. Further, by Lemma 5.12 (1), D' is $\varphi(v_0)$-reduced. By the induction hypothesis, we can take a point $w' \in \Gamma_{/B}$ such that

$$E' := D' + [w'] \in \mathrm{Div}(\Gamma_{/B})$$

is $\varphi(v_0)$-reduced.

We claim that $w' \neq v'_\infty$. Indeed, since $D|_B$ satisfies the conditions (i)(ii) in Lemma 5.10 and since $\deg(D|_B) = g(B)$, we can take a point $v \in B \setminus V(B)$ such that $D+[v]$ satisfies the conditions (i)(ii) in Lemma 5.10. We have $\deg(D+[v])|_B = \deg(D|_B)+1 = g(B)+1$. By the above assumption, $D+[v]$ is not v_0-reduced on Γ, and thus $D'+[v'_\infty]$ is not v'_0-reduced by Lemma 5.12(2). We have shown the claim.

Since $w' \neq v'_\infty$, we set $w = \varphi^{-1}(w')$. Note that $w \notin B$. We let

$$E := D + [w] \in \mathrm{Div}(\Gamma).$$

We will show that E is v_0-reduced. Let $A \subseteq \Gamma \setminus \{v_0\}$ be a non-empty compact connected subset. We will show that ∂A has a non-saturated point for E.

Since D is v_0-reduced, it suffices to consider the case $w \in \partial A$, so that we may assume that $A \not\subseteq B$. We set

$$A' = \varphi(A).$$

Since $D' + [w']$ is $\varphi(v_0)$-reduced, there exists a non-saturated point $a' \in \partial A'$ for $D' + [w']$.

Subcase 3-1. Suppose that A' contains v'_∞ as an interior point. Then $v'_\infty \notin \partial A'$, so that $a' \neq v'_\infty$. Then $a := \varphi^{-1}(a') \in \partial A$, and a is a non-saturated point for E. Thus ∂A has a non-saturated point for E.

Subcase 3-2. Suppose that $A' \not\ni v'_\infty$. Again, $v'_\infty \notin \partial A'$, so that ∂A has a non-saturated point $a := \varphi^{-1}(a')$ for E.

Now we consider the complementary case to Subcases 3-1 and 3-2, so that we assume that A' has v'_∞ as a boundary point.

We write $\partial B = \{v_B, \iota(v_B)\}$. Since B is a v_0-special binary subgraph and Γ is not a binary metric graph, there exists an edge m_B with an endpoint v_B such that $m_B \not\subseteq B$. Then $\iota(m_B)$ is an edge with an endpoint $\iota(v_B)$ such that $\iota(m_B) \not\subseteq B$. Interchanging v_B and $\iota(v_B)$ if necessary, we may assume that $\varphi(\iota(m_B)) \cap U = \{v'_\infty\}$, where U is a small neighborhood of v'_∞. Then $A \cap m_B \neq \{v'_\infty\}$, so that $v_B \in A$.

Subcase 3-3. Suppose that A' has v'_∞ as a boundary point and that $D(\iota(v_B)) = 0$.

Suppose first that $\iota(v_B) \in A$. By our choice of $\iota(v_B)$, we have $\iota(v_B) \in \partial A$. Also we have $\mathrm{outdeg}_A^\Gamma(\iota(v_B)) \geq 1$ (contribution from the direction of $\iota(m_B)$). Thus $\iota(V_B) \in \partial A$ is a non-saturated point for E.

Suppose next that $\iota(v_B) \notin A$. We consider $A \cap B$. Since $v'_\infty \in A'$, $A \cap B$ is a non-empty compact subset of B with $\iota(v_B) \notin A \cap B$. Since $\deg(E|_B) = \deg(D|_B) = g(B)$, Corollary 5.7 tells us that there exists a non-saturated point $b \in \partial^B(A \cap B)$ for $D|_B$ on B. Then $b \in \partial A$, and b is a non-saturated point for D on Γ. Since $w \notin B$, $b \in \partial A$ is also non-saturated for E. We conclude that ∂A has a non-saturated point b for E.

Subcase 3-4. Finally, suppose that A' has v'_∞ as a boundary point and that $D(\iota(v_B)) > 0$. We will show that ∂A has a non-saturated for E.

In this case, since $D|_B$ satisfies the conditions (i)(ii) of Lemma 5.10, we have $D(v_B) = 0$.

Since $\deg(D|_B) = g(B)$ and $D|_B$ satisfies the conditions (i)(ii) of Lemma 5.10, there exists an edge e_1 with $\mathrm{Supp}(D) \cap \mathring{e}_1 = \emptyset$. We take $v_1 \in \mathring{e}_1$. By Assumption 5.13.1, there exists a non-empty compact connected subset $A_1 \subseteq \Gamma \setminus \{v_0\}$ such that any point in ∂A_1 is saturated for $D + [v_1]$. Since D is v_0-reduced, this means that v_1 is the only point in ∂A_1 that is not saturated for D. Since $E = D + [w]$, if a point $b \in \partial A_1$ is non-saturated for E, then $b = v_1$.

We will show that $\iota(v_B) \in \partial A_1$ and $\iota(v_B)$ is an interior point of $A_1 \cup B$. To show that $\iota(v_B) \in \partial A_1$, we consider an auxiliary divisor $F := D + [v_1] - D(\iota(v_B))[\iota(v_B)] \in \mathrm{Div}(\Gamma)$. Since $D(\iota(v_B)) > 0$,

we see that F is an effective divisor, $\deg(F|_B) \leq \deg(D|_B) = g(B)$ and F satisfies the conditions (i)(ii) of Lemma 5.10. By Lemma 5.14 which we show below, we have $A_1 \cap B \neq B$. Further, since $v_1 \in (\partial A_1) \cap B$, we have $A_1 \cap B \neq \emptyset$. Therefore, by Corollary 5.7, there exists a point $a_1 \in \partial^B(A_1 \cap B)$ with $\mathrm{outdeg}^B_{A_1 \cap B}(a_1) > F(a_1)$. We note that $a_1 \in \partial^B(A_1 \cap B) \subseteq \partial A_1$. Since any point in ∂A_1 is saturated for $F + D(\iota(v_B))[\iota(v_B)] = D + [v_1]$, we have $\mathrm{outdeg}^\Gamma_{A_1}(a_1) \leq (F + D(\iota(v_B))[\iota(v_B)])(a_1)$. Thus

$$F(a_1) < \mathrm{outdeg}^B_{A_1 \cap B}(a_1) \leq \mathrm{outdeg}^\Gamma_{A_1}(a_1) \leq (F + D(\iota(v_B))[\iota(v_B)])(a_1).$$

It follows that $\iota(v_B) = a_1$, so that $\iota(v_B) \in \partial^B(A_1 \cap B) \subseteq \partial A_1$.

Next we show that $\iota(v_B)$ is an interior point of $A_1 \cup B$. To see this, we first note that, since $\partial B \not\subseteq A_1 \cap B$ (cf. Lemma 5.14) and $\iota(v_B) \in \partial A_1$, we have $v_B \notin A_1$. We consider an auxiliary divisor $F' := D + [v_1] - [\iota(v_B)] \in \mathrm{Div}(\Gamma)$. Since $D(\iota(v_B)) > 0$, we see that F' is an effective divisor, $\deg(F'|_B) = g(B)$, F' satisfies the conditions (i)(ii) of Lemma 5.10, and $F'(v_B) = D(v_B) = 0$. Then Corollary 5.7 (with v' being v_B), there exists a point $a'_1 \in \partial^B(A_1 \cap B)$ with $\mathrm{outdeg}^B_{A_1 \cap B}(a'_1) > F'(a'_1)$. Arguing similarly as above, we see that $a'_1 = \iota(v_B)$, and this time, the estimate we obtain is

$$F'(\iota(v_B)) < \mathrm{outdeg}^B_{A_1 \cap B}(\iota(v_B)) \leq \mathrm{outdeg}^\Gamma_{A_1}(\iota(v_B)) \leq (F' + [\iota(v_B)])(\iota(v_B)) = F'(\iota(v_B)) + 1.$$

It follows that $\mathrm{outdeg}^B_{A_1 \cap B}(\iota(v_B)) = \mathrm{outdeg}^\Gamma_{A_1}(\iota(v_B))$. Thus $\iota(v_B)$ is an interior point of $A_1 \cup B$.

We put $A_2 := A_1 \cup A \cup B$. Since $v_B \in A$ and $\iota(v_B) \in A_1$, we have $\partial B \subseteq A_1 \cup A$ and $\partial A_2 \subseteq \partial(A_1 \cup A)$. Since $\iota(v_B)$ is an interior point of $A_1 \cup B$, one has $\iota(v_B) \in A_2 \setminus \partial A_2$.

Since $E(v_B) = D(v_B) = 0$, if v_B is not an interior point of A, then $v_B \in \partial A$ is non-saturated for E. Thus we may assume that v_B is an interior point of A. It follows that $v_B \in A_2 \setminus \partial A_2$, so that $\partial B \subseteq A_2 \setminus \partial A_2$. Since any point of $B \setminus \partial B$ is an interior point of A_2, A_2 contains a neighborhood of B and thus $(\partial A_2) \cap B = \emptyset$. We put $A'_2 := \varphi(A_2)$. Since $E' = D' + [w']$ is $\varphi(v_0)$-reduced, there exists a non-saturated point $a'' \in \partial A'_2$ for E'. Since $(\partial A_2) \cap B = \emptyset$, there exists a unique $\widetilde{a} \in \Gamma \setminus B$ with $\varphi(\widetilde{a}) = a''$. Then $\widetilde{a} \in \partial A_2$ and $\widetilde{a}$ is non-saturated for E (with respect to A_2).

We claim that $\widetilde{a} \in \partial A$. Indeed, suppose that $\widetilde{a} \notin \partial A$. Then, since $\partial A_2 \subseteq \partial A_1 \cup \partial A$, we have $\widetilde{a} \in \partial A_1$. Since $A_1 \subseteq A_2$, we will then have

$$\mathrm{outdeg}^\Gamma_{A_1}(\widetilde{a}) \geq \mathrm{outdeg}^\Gamma_{A_2}(\widetilde{a}) > E(\widetilde{a}),$$

which shows that $\widetilde{a} \in \Gamma \setminus B$ is a non-saturated point for E (with respect to A_1). This contradicts the fact that $v_1 \in B$ is the only possible point in ∂A_1 that is non-saturated for E (with respect to A_1). Thus $\widetilde{a} \in \partial A$, and ∂A has a non-saturated point $\widetilde{a}$ for $E = D + [w]$.

This completes the proof of the Case 3, and also the proof of Theorem 5.1. □

Lemma 5.14. *In Subcase 3.4, we have $\partial B \not\subseteq A_1$.*

Proof. First we show $\partial B \not\subseteq A_1$. To derive a contradiction, suppose that $\partial B \subseteq A_1$. Then we have $\partial(A_1 \cup B) \subseteq \partial A_1$. Since any point in ∂A_1 is saturated for $D + [v_1]$, it follows that any point in $\partial(A_1 \cup B)$ is saturated for $D + [v_1]$. On the other hand, $v_1 \in \mathring{e}_1$ implies $v_1 \notin \partial(A_1 \cup B)$. Thus any point in $\partial(A_1 \cup B)$ is saturated for D. This contradicts with our assumption that D is v_0-reduced. We conclude that $\partial B \not\subseteq A_1$. □

We have the following corollaries of Theorem 5.1, which will be needed to prove Theorem 1.11.

Corollary 5.15. *Let Γ be as in Theorem 1.10. Let v_0 be an element of Γ with $\iota(v_0) = v_0$. Let D a v_0-reduced divisor on Γ. Assume that $p_\Gamma(D) = 0$ and $\deg(D) \leq g - 1$. Then there exists a divisor E on Γ such that*

$$D \leq E, \quad \deg(E) = g, \quad p_\Gamma(E) = 0.$$

Proof. Since $p_\Gamma(D) = 0$, we have $D(v_0) \leq 1$.

Case 1. Assume that $D_0(v_0) = 0$. Using Theorem 1.10 repeatedly, there exist $w_{\deg(D)+1}, \ldots, w_g \in \Gamma \setminus \{v_0\}$ such that $E := D + [w_{\deg(D)+1}] + \cdots + [w_g]$ is v_0-reduced. If $p_\Gamma(E) \geq 1$, then $E - 2[v_0]$

is linearly equivalent to an effective divisor. However, $E - 2[v_0]$ is v_0-reduced and the coefficient at v_0 is -2, this is impossible. Thus we get $p_\Gamma(E) = 0$.

Case 2. Assume that $D(v_0) = 1$. We put $D' = D - [v_0]$. Then D' is v_0-reduced, and using Theorem 1.10 repeatedly, there exist $w_{\deg(D')+1}, \ldots, w_{g-1} \in \Gamma \smallsetminus \{v_0\}$ such that $E' := D' + [w_{\deg(D')+1}] + \cdots + [w_{g-1}]$ is v_0-reduced. Put $E = E' + [v_0]$. Then E is v_0-reduced, and as in Case 1 we have $D \leq E$, $\deg(E) = g$ and $p_\Gamma(E) = 0$. □

Corollary 5.16. *Let Γ be as in Theorem 1.10. Let D be an effective divisor on Γ. Assume that $p_\Gamma(D) = 0$ and $\deg(D) = g$. Then $r_\Gamma(D) = 0$.*

Proof. Recall that we have fixed a point v_0 on Γ satisfying (3.2). Let D_0 be the v_0-reduced divisor on Γ which is linearly equivalent to D. Then D_0 is effective. Since $p_\Gamma(D) = 0$, we have $D_0(v_0) \leq 1$. We may and do replace D_0 with D.

Case 1. Assume that $D(v_0) = 0$. In this case, $D - [v_0]$ is also v_0-reduced, so that $D - [v_0]$ is not linearly equivalent to an effective divisor. Thus $r_\Gamma(D - [v_0]) = -1$. Hence $r_\Gamma(D) \leq 0$. Since D is effective, we have $r_\Gamma(D) = 0$.

Case 2. Assume that $D(v_0) = 1$. We set $D' = D - [v_0]$. Then D' is effective and v_0-reduced. Since $p_\Gamma(D) = 0$, we have $p_\Gamma(D') = 0$.

By Theorem 1.10, there exists $w \in V(G) \smallsetminus \{v_0\}$ such that $D' + [w]$ is v_0-reduced.

To derive a contradiction, we assume that $r_\Gamma(D) \neq 0$. Since D' is v_0-reduced and $D'(v_0) = 0$, we have $r_\Gamma(D') = 0$. Thus $r_\Gamma(D) \leq 1$. It follows that $r_\Gamma(D) = 1$.

Since $r_\Gamma(D) = 1$, $D - [\iota(w)]$ is linearly equivalent to an effective divisor D''. By Theorem 2.5(2), we may assume that D'' is v_0-reduced. Then $D'' + [v_0]$ is v_0-reduced.

On the other hand, we have

$$\begin{aligned} D'' + [v_0] &\sim D - [\iota(w)] + [v_0] \sim (D' + [v_0]) - [\iota(w)] + [v_0] \\ &\sim D' + 2[v_0] - [\iota(w)] \sim D' + ([w] + [\iota(w)]) - [\iota(w)] \sim D' + [w]. \end{aligned}$$

Since w is taken so that $D' + [w]$ is v_0-reduced, the uniqueness of v_0-reduced divisors (Theorem 2.5(1)) implies that $D'' + [v_0] = D' + [w]$ in $\mathrm{Div}(\Gamma)$. However, the coefficient of $D'' + [v_0]$ at v_0 is at least 1, while that of $D' + [w]$ is 0. This is a contradiction. It follows that $r_\Gamma(D) = 0$. □

6. Ranks of divisors on a hyperelliptic graph

In this section, we prove Theorem 1.11. We need the following approximation result. Recall that $p_\Gamma(D)$ is defined in (3.3).

Proposition 6.1. *Let G be a finite graph and Γ the metric graph associated to G. Assume that Γ is hyperelliptic. Let $D = \sum_{i=1}^n a_i[v_i]$ be an effective divisor on Γ. Then there exist $v_1', \ldots, v_i' \in \Gamma_{\mathbb{Q}}$ such that $D' := \sum_{i=1}^n a_i[v_i']$ satisfies*

$$r_\Gamma(D) = r_\Gamma(D') \quad \textit{and} \quad p_\Gamma(D) = p_\Gamma(D'). \tag{6.8}$$

Proof. The existence of D' satisfying the first condition $r_\Gamma(D) = r_\Gamma(D')$ is proved in Gathmann–Kerber [15, the proof of Proposition 3.1] (where D is not necessarily effective), and the existence of D' satisfying (6.8) is shown by a similar argument. For the reader's convenience, we give a brief proof. We set

$$M := \Gamma^n = \left\{ (v_1', \ldots, v_n') \;\middle|\; v_1', \ldots, v_n' \in \Gamma \right\}.$$

To $(v_1', \ldots, v_n') \in M$, we attach the divisor $D' = \sum_{i=1}^n a_i[v_i'] \in \mathrm{Div}(\Gamma)$. Gathmann–Kerber [15, the proof of Proposition 3.1] shows that the locus D' in M such that $r_\Gamma(D') < r_\Gamma(D) + 1$ is an open neighborhood U of D, and the locus of $r_\Gamma(D') \geq r_\Gamma(D)$ is a union A of closed polyhedra. We set

$d = \deg D$, $r = p_\Gamma(D)$. Let $v_0 \in \Gamma_{\mathbb{Q}}$ be a point as in 3.2. For $j = r, r+1$, we set

$$T_j = \left\{ (v_1', \dots, v_n', f, P_1, \dots, P_{d-2j}) \;\middle|\; \begin{array}{c} v_1', \dots, v_n' \in \Gamma, P_1, \dots, P_{d-2j} \in \Gamma, \\ f \text{ a rational function on } \Gamma, \\ \sum_{i=1}^{n} a_i[v_i'] - 2j[v_0] + (f) = P_1 + \cdots + P_{d-2j} \end{array} \right\}$$

and

$$\pi_j : T_j \to M, \quad (v_1', \dots, v_n', f, P_1, \dots, P_{d-2j}) \mapsto (v_1', \dots, v_n').$$

Note that $p_\Gamma(D') \geq r$ for $D' = \sum_{i=1}^n a_i[v_i']$ if and only if $(v_1', \dots, v_n') \in \pi(T_r)$. By [15, the proof of Proposition 3.1], T_j is a polyhedral complex, and π_j is a morphism of polyhedral complexes. Thus the image $\pi(T_j)$ is a union of closed polyhedra in M. It follows that the locus D' in M such that $p_\Gamma(D') < r+1$ is an open neighborhood V of D, and the locus of $p_\Gamma(D') \geq r$ is a union B of closed polyhedra.

Since $U \cap V \cap A \cap B$ contains D and thus non-empty, and since all polyhedral complexes and morphisms involved in the above construction are defined over $\mathbb{Q}$, there exists a rational point in $U \cap V \cap A \cap B$. It follows that there exists $D' \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ such that $r_\Gamma(D) = r_\Gamma(D')$ and $p_\Gamma(D) = p_\Gamma(D')$. □

We prove Theorem 1.11, using Proposition 6.1, Corollary 5.15, Corollary 5.16, Baker's Specialization Lemma (Theorem 2.8) and computation of ranks of divisors on hyperelliptic curves over a field (which will be proved in Theorem 7.1 below). Recall that $r_{(\Gamma,\omega)}(D)$ and $p_{(\Gamma,\omega)}(D)$ are respectively defined in (2.1) and (3.4).

Theorem (= Theorem 1.11)**.** *Let (G, ω) be a hyperelliptic vertex-weighted graph, and Γ the metric graph associated to G. Set $g = g(\Gamma, \omega)$. Let D be an effective divisor on Γ. Then*

$$r_{(\Gamma,\omega)}(D) = \begin{cases} p_{(\Gamma,\omega)}(D) & (\text{if } \deg(D) - p_{(\Gamma,\omega)}(D) \leq g), \\ \deg(D) - g & (\text{if } \deg(D) - p_{(\Gamma,\omega)}(D) \geq g+1). \end{cases}$$

Proof. **Step 1.** Let G^ω be the virtual weightless graph of (G, ω), and let Γ^ω be the virtual weightless metric graph of (G, ω). Note that Γ^ω is the metric graph associated to G^ω. By Proposition 3.11, Γ^ω is a hyperelliptic graph. Let $\jmath : \Gamma \hookrightarrow \Gamma^\omega$ be the natural embedding. Since $g(\Gamma, \omega) = g(\Gamma^\omega)$, $r_{(\Gamma,\omega)}(D) = r_{\Gamma^\omega}(D)$ and $p_{(\Gamma,\omega)}(D) = p_{\Gamma^\omega}(D)$, it suffices to prove the Theorem for the weightless case, i.e., for G^ω and Γ^ω.

Step 2. By Step 1, we replace G^ω by G, and Γ^ω by Γ. Let $\overline{\Gamma}$ be the metric graph obtained by contracting all bridges of Γ, and $\rho : \Gamma \to \overline{\Gamma}$ the retraction map. Since $r_\Gamma(D) = r_{\overline{\Gamma}}(\rho_*(D))$ by Lemma 2.10(2), and since $p_\Gamma(D) = p_{\overline{\Gamma}}(\rho_*(D))$ by Lemma 3.14(3) for any divisor D on Γ, we may and do assume that Γ has no points of valence 1. Then Γ has a unique hyperelliptic involution ι (cf. Theorem 3.5).

Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field such that $\mathrm{char}(k) \neq 2$. We take a regular, semi-stable R-curve $\mathscr{X}$ such that the reduction graph of $\mathscr{X}$ is $G = (G, \mathbf{0})$ as in Theorem 4.6. In particular, the generic fiber X of $\mathscr{X}$ is hyperelliptic. Let $\iota_X : X \to X$ be the hyperelliptic involution of X. We take a Weierstrass point $P_0 \in X(\overline{\mathbb{K}})$, i.e., $\iota_X(P_0) = P_0$. We set $v_0 = \tau(P_0) \in \Gamma_{\mathbb{Q}}$. Then we have $\iota(v_0) = v_0$ by Theorem 4.6.

Let D be an effective divisor on Γ. By Proposition 6.1, we may and do assume that $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$. Let D_0 be the v_0-reduced divisor linearly equivalent to D. Then we have $D_0 \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ by Theorem 2.5(3). We set $r = \left\lfloor \frac{D_0(v_0)}{2} \right\rfloor$ and $s = \deg(D) - 2r$. Then D_0 is written as

$$D_0 = 2r[v_0] + [w_1] + \cdots + [w_s]$$

for some $w_1, \dots, w_s \in \Gamma_{\mathbb{Q}}$. If $\iota(w_i) = w_j$ for some $i \neq j$, then $[w_i] + [w_j] \sim 2[v_0]$ by Lemma 3.9, and $D_0 \sim 2(r+1)[v_0] + \sum_{k=1, k\neq i,j}^{s} [w_k]$. This contradicts with the fact that D_0 is v_0-reduced. Hence we have $p_\Gamma(D) = r$ and $\iota(w_i) \neq w_j$ for any $i \neq j$. Also, $p_\Gamma([w_1] + \cdots + [w_s]) = 0$.

By Proposition 2.7(1), we take $Q_1, \ldots, Q_s \in X(\overline{\mathbb{K}})$ such that $\tau(Q_i) = w_i$ for $i = 1, \ldots, s$. Since $\iota(\tau(Q)) = \tau(\iota_X(Q))$ for any $Q \in X(\overline{K})$ by Theorem 4.6, we have $\iota_X(Q_i) \neq Q_j$ for $i \neq j$.

Case 1. Assume that $\deg(D) - p_\Gamma(D) \leq g$. Note that $s \leq r + s = \deg(D) - r \leq g$. Since $[w_1] + \cdots + [w_s]$ is v_0-reduced and $p_\Gamma([w_1] + \cdots + [w_s]) = 0$, Corollary 5.15 tells us that there exist $w_{s+1}, \ldots, w_g \in \Gamma$ with $p_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g]) = 0$. By Corollary 5.16, we have $r_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g]) = 0$. Hence $r_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_{s+r}]) = 0$. Since $2v_0 \sim v + \iota(v)$ for any $v \in \Gamma$ by Lemma 3.9, we have

$$\begin{aligned} D &\sim 2r[v_0] + [w_1] + \cdots + [w_s] \\ &\sim [w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_{s+r}] + [\iota(w_{s+1})] + \cdots + [\iota(w_{s+r})]. \end{aligned}$$

Since $r_\Gamma(E) \leq r_\Gamma(E - [v]) + 1$ for any divisor E and $v \in \Gamma$, we have

$$r_\Gamma(D) \leq r_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_{s+r}]) + r = r. \tag{6.9}$$

On the other hand, for any $u_1, \ldots, u_r \in \Gamma$, we have

$$\begin{aligned} D - (u_1 + \cdots + u_r) &\sim 2r[v_0] - (u_1 + \cdots + u_r) + [w_1] + \cdots + [w_s] \\ &\sim \iota(u_1) + \cdots + \iota(u_r) + [w_1] + \cdots + [w_s] \end{aligned}$$

by Lemma 3.9. This shows $r_\Gamma(D) \geq r$. Thus we conclude that $r_\Gamma(D) = r$, which is the desired estimate when $\deg(D) - p_\Gamma(D) \leq g$.

Case 2. Assume that $\deg(D) - p_\Gamma(D) \geq g + 1$.

Subcase 2-1. Assume that $s \leq g$. Since $[w_1]+\cdots+[w_s]$ is v_0-reduced and $p_\Gamma([w_1]+\cdots+[w_s]) = 0$, Corollary 5.15 tells us that there exist $w_{s+1}, \ldots, w_g \in \Gamma$ with $p_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g]) = 0$. By Proposition 6.1, $w_{s+1}, \ldots, w_g$ are taken as elements of $\Gamma_{\mathbb{Q}}$. By Corollary 5.16, we have $r_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g]) = 0$. Recalling that $2[v_0] \sim [v] + [\iota(v)]$ for any $v \in \Gamma$ by Lemma 3.9 and that $r + s = \deg(D) - p_\Gamma(D) \geq g + 1$, we have

$$\begin{aligned} D &\sim 2r[v_0] + [w_1] + \cdots + [w_s] \\ &\sim 2(r + s - g)[v_0] + [w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g] + [\iota(w_{s+1})] + \cdots + [\iota(w_g)]. \end{aligned}$$

As in Eqn. (6.9), we have

$$\begin{aligned} r_\Gamma(D) &\leq r_\Gamma([w_1] + \cdots + [w_s] + [w_{s+1}] + \cdots + [w_g]) + 2r + s - g \\ &= 2r + s - g = \deg(D) - g. \end{aligned}$$

By Proposition 2.7(1), we take $P_{s+1}, \ldots, P_g \in X(\overline{\mathbb{K}})$ such that $\tau(P_i) = w_i$ for $i = s + 1, \ldots, g$. By Theorem 4.6, we have $\tau(\iota_X(P_i)) = \iota(w_i)$ for $i = s + 1 \ldots, g$.

We put

$$\widetilde{D} = 2(r + s - g)P_0 + Q_1 + \cdots + Q_s + P_{s+1} + \cdots + P_g + \iota(P_{s+1}) + \cdots + \iota(P_g).$$

By Theorem 7.1 which we will prove in the next section, we then have $r_X(\widetilde{D}) = \deg(\widetilde{D}) - g$. Since $\tau_*(\widetilde{D}) = D$, the specialization lemma (Theorem 2.8) gives

$$r_\Gamma(D) \geq r_X(\widetilde{D}) = \deg(\widetilde{D}) - g = \deg(D) - g.$$

We conclude that $r_\Gamma(D) = \deg(D) - g$, which is the desired estimate.

Subcase 2-2. Assume that $s \geq g + 1$. Since $p_\Gamma([w_1] + \cdots + [w_s]) = 0$, we have $p_\Gamma([w_1] + \cdots + [w_g]) = 0$. By Corollary 5.16, we have $r_\Gamma([w_1] + \cdots + [w_g]) = 0$. As in Eqn. (6.9), we have

$$r_\Gamma(D) \leq r_\Gamma([w_1] + \cdots + [w_g]) + 2r + s - g = 2r + s - g = \deg(D) - g.$$

We put $\widetilde{D} = 2rP_0 + Q_1 + \cdots + Q_s$. As in Subcase 2-1, Theorem 7.1 gives $r_X(\widetilde{D}) = \deg(\widetilde{D}) - g$. We argue as in Subcase 2-1 to conclude $r_\Gamma(D) = \deg(D) - g$. This completes the proof of Theorem 1.11. ∎

7. Ranks of divisors on a hyperelliptic curve

Let Ω be an algebraically closed field. Let X be a smooth connected projective curve over Ω. Recall that X is a hyperelliptic curve if the genus of X is at least 2 and there exists a divisor D on X such that $\deg(D) = 2$ and $r_X(D) = 1$ (see Example 4.2). In this section, we prove the following theorem as promised in the previous section.

Theorem 7.1. *Let X be a hyperelliptic curve defined over Ω. We denote the involution of X by ι_X. Let D be an effective divisor on X. We express D as*

$$D = P_1 + \cdots + P_r + \iota_X(P_1) + \cdots + \iota_X(P_r) + Q_1 + \cdots + Q_s,$$

where $P_1, \ldots, P_r, Q_1, \ldots, Q_s \in X(\Omega)$ and $\iota_X(Q_i) \neq Q_j$ for any $i \neq j$ with $1 \leq i, j \leq s$. (This expression is unique up to a transposition of P_i with $\iota_X(P_i)$, and permutations of $\{P_1, \ldots, P_r\}$ and $\{Q_1, \ldots, Q_s\}$.) Then we have

$$r_X(D) = \begin{cases} r & (\text{if } \deg(D) - r \leq g), \\ \deg(D) - g & (\text{if } \deg(D) - r \geq g + 1). \end{cases}$$

First we prove a particular case of Theorem 7.1.

Proposition 7.2. *Under the notation in Theorem 7.1, let $Q_1, \ldots, Q_g$ be points on $X(\Omega)$ such that $\iota_X(Q_i) \neq Q_j$ for any $i \neq j$ with $1 \leq i, j \leq g$. Then $r_X(Q_1 + \cdots + Q_g) = 0$.*

Proof. We recall that, for a divisor D on X, the rank $r_X(D)$ of D is equal to $h^0(X, \mathcal{O}_X(D)) - 1$. Let $\iota_X : X \to X$ denote the unique involution such that X/ι_X is isomorphic to $\mathbb{P}^1$. Fixing $X/\iota_X \cong \mathbb{P}^1$, we set $\pi : X \to X/\iota_X \cong \mathbb{P}^1$.

We put $E := Q_1 + \cdots + Q_g$ and $L := \mathcal{O}_X(E)$. Let $\varphi : \mathcal{O}_{\mathbb{P}^1} \to \pi_* L$ be the composition of the canonical homomorphisms $\mathcal{O}_{\mathbb{P}^1} \to \pi_* \mathcal{O}_X$ with $\pi_* \mathcal{O}_X \to \pi_* L$. For a point $P \in \mathbb{P}^1(\Omega)$, let $\psi_P : \mathcal{O}_{\pi^{-1}(P)} \to \mathcal{O}_X(E)|_{\pi^{-1}(P)}$ be the homomorphism given by the restriction of the canonical homomorphism $\mathcal{O}_X \to \mathcal{O}_X(E)$ to $\pi^{-1}(P)$. Then we have the commutative diagram

$$\begin{array}{ccccccc} \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{\varphi} & \pi_* L & \longrightarrow & \pi_* L \otimes \mathcal{O}_P & \xrightarrow{\cong} & \mathcal{O}_X(E)|_{\pi^{-1}(P)} \\ \| & & \uparrow & & \uparrow & & \psi_P \uparrow \\ \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & \pi_* \mathcal{O}_X & \longrightarrow & \pi_* \mathcal{O}_X \otimes \mathcal{O}_P & \xrightarrow{\cong} & \mathcal{O}_{\pi^{-1}(P)}, \end{array}$$

where all divisors on $\mathbb{P}^1$ and X are naturally regarded as subschemes of $\mathbb{P}^1$ and X, respectively.

We claim that $\psi_P(1) \neq 0$ for any $P \in \mathbb{P}^1(\Omega)$. Indeed, the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_E(E) \to 0$$

gives rise to the exact sequence

$$\mathcal{O}_{\pi^{-1}(P)} \xrightarrow{\psi_P} \mathcal{O}_X(E)|_{\pi^{-1}(P)} \xrightarrow{\phi_P} \mathcal{O}_E(E) \otimes \mathcal{O}_{\pi^{-1}(P)} \longrightarrow 0. \tag{7.10}$$

Let $\mathcal{O}_{E \cap \pi^{-1}(P)}$ denotes the structure sheaf of the scheme-theoretic intersection $E \cap \pi^{-1}(P)$. Then $\mathcal{O}_E \otimes \mathcal{O}_{\pi^{-1}(P)} = \mathcal{O}_{E \cap \pi^{-1}(P)}$.

Our assumption on $Q_1, \ldots, Q_g$ tells us that $\pi^{-1}(P) \nsubseteq E$ as subschemes of X, so that

$$\dim_\Omega \mathcal{O}_E(E) \otimes \mathcal{O}_{\pi^{-1}(P)} < \dim_\Omega \mathcal{O}_{\pi^{-1}(P)} = \dim_\Omega \mathcal{O}_X(E)|_{\pi^{-1}(P)}.$$

It follows that ϕ_P is not injective. Since the exact sequence (7.10) that the image of ψ_P is non-trivial, we conclude $\psi_P(1) \neq 0$.

We consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\varphi} \pi_* L \to M \to 0. \tag{7.11}$$

Since $\pi_* L$ is a vector bundle of rank 2, M has rank 1. Since $\psi_P(1) \neq 0$ for any $P \in \mathbb{P}^1$, $\varphi(1)(P) \neq 0$ at any $P \in \mathbb{P}^1$ and thus M is a line bundle. Since π is an affine map and $\deg(L) = g$, using the Riemann–Roch formulae for $\pi_* L$ and L, we obtain

$$\deg(\pi_* L) = \chi(\mathbb{P}^1, \pi_* L) - 2 = \chi(X, L) - 2 = (1 - g + \deg(L)) - 2 = -1.$$

It follows that $\deg M = \deg \pi_* L = -1$ and, in particular, $h^0(\mathbb{P}^1, M) = 0$. Taking the cohomology of (7.11), we then obtain $h^0(\mathbb{P}^1, \pi_* L) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1$. Since π is an affine map, we have

$$r_X(E) = h^0(X, L) - 1 = h^0(\mathbb{P}^1, \pi_* L) - 1 = 0.$$

This completes the proof. □

Proof of Theorem 7.1. We prove Theorem 7.1 using Proposition 7.2.

Case 1. Suppose that $\deg(D) - r = s + r \leq g$. We take $Q_{s+1}, \ldots, Q_g \in X(\Omega)$ such that $\iota_X(Q_i) \neq Q_j$ for any $i \neq j$ with $1 \leq i, j \leq g$ Since $P + \iota_X(P) \sim Q + \iota_X(Q)$ for any $P, Q \in X(\Omega)$, we have

$$D \sim Q_{s+1} + \cdots + Q_{s+r} + \iota_X(Q_{s+1}) + \cdots + \iota_X(Q_{s+r}) + Q_1 + \cdots + Q_s.$$

Then $r_X(D) \leq r_X(Q_1 + \cdots + Q_s + Q_{s+1} + \cdots + Q_{s+r}) + r$. By Proposition 7.2, we have $0 \leq r_X(Q_1 + \cdots + Q_{s+r}) \leq r_X(Q_1 + \cdots + Q_g) = 0$. Thus we get $r_X(D) \leq r$. On the other hand, for any $R_1, \ldots, R_r \in X(\Omega)$, we have

$$D - (R_1 + \ldots + R_r) \sim \iota_X(R_1) + \cdots + \iota_X(R_r) + Q_1 + \cdots + Q_s.$$

Thus $r_X(D) \geq r$. We conclude that $r_X(D) = r$, which gives the desired equality when $s + r \leq g$.

Case 2. Suppose that $\deg(D) - r = s + r \geq g + 1$. We take $Q_{s+1}, \ldots, Q_{s+r} \in X(\Omega)$ such that $\iota_X(Q_i) \neq Q_j$ for any $i \neq j$ with $1 \leq i, j \leq s + r$. As in Case 1, we have

$$D \sim Q_{s+1} + \cdots + Q_{s+r} + \iota_X(Q_{s+1}) + \cdots + \iota_X(Q_{s+r}) + Q_1 + \cdots + Q_s.$$

Then $r_X(D) \leq r_X(Q_1 + \cdots + Q_s + Q_{s+1} + \cdots + Q_g) + (s + 2r - g)$. By Proposition 7.2, we have $r_X(D) \leq s + 2r - g = \deg(D) - g$. On the other hand, the Riemann–Roch formula on X gives

$$r_X(D) = r_X(K_X - D) + 1 - g + \deg(D) \geq -g + \deg(D).$$

We conclude that $r_X(D) = \deg(D) - g$, which gives the desired equality when $s + r \geq g + 1$. □

8. Proofs of Theorem 1.2 and Proposition 1.4

In this section, we prove Theorem 1.2 and Proposition 1.4 and give several examples. We also consider Question 1.1 for a vertex-weighted graph of genus 0 or 1.

We begin by proving Theorem 1.2.

Lemma 8.1. *The condition* (ii) *implies* (i) *in Theorem 1.2.*

Proof. Let (G, ω) be a hyperelliptic vertex-weighted graph and Γ the metric graph associated to G. By definition, there exists a divisor $D \in \mathrm{Div}(\Gamma)$ such that $\deg(D) = 2$ and $r_{(\Gamma,\omega)}(D) = 1$. In view of [15, Proposition 3.1] (cf. Proposition 6.1), D is taken in $\mathrm{Div}(\Gamma_{\mathbb{Q}})$. Assuming (ii), there exist a regular, semi-stable R-curve $\mathscr{X}$ with reduction graph (G, ω) and $\widetilde{D} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $D = \tau_*(\widetilde{D})$ and $r_{(\Gamma,\omega)}(D) = r_X(\widetilde{D})$. (Here X is the generic fiber of $\mathscr{X}$ and τ is the specialization map.) It follows that X is a hyperelliptic curve. Then Theorem 1.9 tells us that (G, ω) satisfies the condition (i). □

We show that the condition (C') is not weaker than the condition (C) in the introduction.

Lemma 8.2. *Let (G, ω) be a finite vertex-weighted graph, and let Γ the metric graph associated to G. Assume that there exists a regular, semi-stable R-curve $\mathscr{X}$ with the reduction graph (G, ω) satisfying the condition* (C'). *Then $\mathscr{X}$ satisfies the condition* (C).

Proof. Let $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$. From the condition (C'), we infer that there exist a divisor $E \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ and $\widetilde{E} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $\tau_*(\widetilde{E}) = E$ and $r_{(\Gamma,\omega)}(E) = r_X(\widetilde{X})$. By [6, Corollary A.9] for metric graphs, the restriction of the specialization map $\tau_*|_{\mathrm{Prin}(X_{\overline{\mathbb{K}}})} : \mathrm{Prin}(X_{\overline{\mathbb{K}}}) \to \mathrm{Prin}(\Gamma_{\mathbb{Q}})$ is surjective, where $\mathrm{Prin}(\Gamma_{\mathbb{Q}}) := \mathrm{Div}(\Gamma_{\mathbb{Q}}) \cap \mathrm{Prin}(\Gamma)$. Since $D - E \in \mathrm{Prin}(\Gamma_{\mathbb{Q}})$, there exists a principal divisor $\widetilde{N}$ such that $\tau_*(\widetilde{N}) = D - E$. We set $\widetilde{D} = \widetilde{E} + \widetilde{N} \in \mathrm{Div}(X_{\mathbb{K}})$. Then $\widetilde{D}$ satisfies $D = \tau_*(\widetilde{D})$ and $r_{(\Gamma,\omega)}(D) = r_X(\widetilde{D})$. □

By Lemma 8.2, (iii) implies (ii) in Theorem 1.2. Thus we need to show that (i) implies (iii) in Theorem 1.2, which amounts to the following.

Theorem 8.3. *Let (G, ω) be a hyperelliptic vertex-weighted graph such that every vertex v of G has at most $(2\omega(v) + 2)$ positive-type bridges emanating from v. Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k with* $\mathrm{char}(k) \neq 2$. *Then there exists a regular, semi-stable R-curve $\mathscr{X}$ with generic fiber X and reduction graph (G, ω) which satisfies the following condition*: *Let Γ be the metric graph associated to G. For any $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, there exist a divisor $E = \sum_{i=1}^{k} n_{v_i}[v_i] \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ that is linearly equivalent to D and $\widetilde{E} = \sum_{i=1}^{k} n_{v_i} P_i \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $\tau(P_i) = v_i$ for any $1 \leq i \leq k$ and $r_{(\Gamma,\omega)}(E) = r_X(\widetilde{E})$.*

Proof. Let $g \geq 2$ denote the genus of (G, ω). If e is a leaf edge with a leaf end v with $\omega(v) = 0$, then we contract e. Let G' be the graph obtained by successively contracting such leaf edges. Then G' is a finite graph such that any leaf edge of G' (if exists) has an leaf end v with $\omega(v) > 0$. We note that G' is seen as a subgraph of G. Let (G', ω') be the vertex-weighted graph, where the vertex-weight function is given by the restriction of ω to $V(G')$.

Let Γ' be the metric graph associated to G'. By Proposition 3.12, Γ' has the hyperelliptic involution $\iota' : \Gamma' \to \Gamma'$ (see Definition 3.13). We remark that Γ' is naturally seen as a subset of Γ.

We take a regular, semi-stable R-curve $\mathscr{X}'$ as in Theorem 4.6. In particular, the generic fiber X of $\mathscr{X}'$ is a hyperelliptic curve, and the dual graph of the special fiber is equal to (G', ω). Further, we have $\tau' \circ \iota_X = \iota' \circ \tau'$ for the specialization map $\tau' : X(\overline{\mathbb{K}}) \to \Gamma'$ and the hyperelliptic involution $\iota_X : X \to X$.

We take a Weierstrass point $P_0 \in X(\overline{\mathbb{K}})$ i.e., a point satisfying $\iota_X(P_0) = P_0$, and put $v_0 = \tau(P_0) \in \Gamma'_{\mathbb{Q}}$. Then we have $\iota'(v_0) = v_0$.

As we have seen in the proof of Corollary 4.7, by successively blowing up closed points on the special fiber, we obtain a regular, semi-stable R-curve $\mathscr{X}$ such that the dual graph of the special fiber is equal to (G, ω). We are going to show that $\mathscr{X}$ satisfies the desired properties.

Let $\tau : X(\overline{\mathbb{K}}) \to \Gamma_{\mathbb{Q}}$ be the specialization map defined by $\mathscr{X}$. Since $\Gamma' \subseteq \Gamma$, we regard v_0 as an element of $\Gamma_{\mathbb{Q}}$. Let $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$. Let $E \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ be the v_0-reduced divisor that is linearly equivalent to D on Γ. We write $E = \sum_{i=1}^{k} n_{v_i}[v_i]$, where we put $v_1 := v_0$, and $v_i \in \Gamma_{\mathbb{Q}}$ for $i = 1, \ldots, k$. By Lemma 2.6, we see that $v_i \in \Gamma'_{\mathbb{Q}}$.

Case 1. Suppose that $r_{(\Gamma,\omega)}(E) = -1$. We take a divisor $\widetilde{E}$ with $\tau_*(\widetilde{E}) = E$. We claim that $r_X(\widetilde{E}) = -1$. Indeed, if $r_X(\widetilde{E}) \geq 0$, then there exists an effective divisor $\widetilde{F} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ with $\widetilde{E} \sim \widetilde{F}$. Then $\tau_*(\widetilde{F})$ is an effective divisor on Γ and, by Proposition 2.7, $D = \tau_*(\widetilde{E}) \sim \tau_*(\widetilde{F})$. This contradicts our assumption that $r_{(\Gamma,\omega)}(E) = -1$ by Lemma 2.12. We conclude that $r_X(\widetilde{E}) = -1$.

Case 2. Suppose that $r_{(\Gamma,\omega)}(E) \geq 0$. Then by Lemma 2.12, E is an effective divisor. As in the proof of Theorem 1.11, if we set $r = \left\lfloor \frac{E(v_0)}{2} \right\rfloor$ and $s = \deg(D) - 2r$, then E is written as

$$E = 2r[v_0] + [w_1] + \cdots + [w_s]$$

for some $w_1, \ldots, w_s \in \Gamma_{\mathbb{Q}}$ such that $\iota(w_i) \neq w_j$ for $i \neq j$. Also, we have $p_{(\Gamma,\omega)}(D) = r$.

By Proposition 2.7(1), we take $Q_1, \ldots, Q_s \in X(\overline{\mathbb{K}})$ such that $\tau(Q_i) = w_i$ for $i = 1, \ldots, s$. As in the proof of Theorem 1.11, we have $\iota_X(Q_i) \neq Q_j$ for $i \neq j$. We set $\widetilde{E} = 2rP_0 + Q_1 + \cdots + Q_s \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$.

By Theorem 1.11 and Theorem 7.1, we then have

$$r_{(\Gamma,\omega)}(E) = r_X(\widetilde{E}) = \begin{cases} r & (\text{if } \deg(D) - r \leq g), \\ \deg(D) - g & (\text{if } \deg(D) - r \geq g + 1). \end{cases}$$

Since $r_{(\Gamma,\omega)}(D) = r_{(\Gamma,\omega)}(E)$, this completes the proof. □

Next we consider a vertex-weighted graph of genus 0 or 1.

Proposition 8.4. *Let $\mathbb{K}$ be a complete discrete valuation field with ring of integers R and algebraically closed residue field k with $\mathrm{char}(k) \neq 2$. Let (G, ω) be a vertex-weighted graph of genus 0 or 1, and Γ the metric graph associated to G. Then there exists a regular, semi-stable R-curve $\mathscr{X}$ with generic fiber X and reduction graph G which satisfies the condition* (C') *in Theorem 1.2.*

Proof. **Case 1.** Suppose that $g(G, \omega) = 0$. This means that $\omega = \mathbf{0}$, and G is a loopless finite graph of genus 0. There exists a regular, strongly semi-stable, totally degenerate R-curve $\mathscr{X}$ with reduction graph G. Let X denote the generic fiber of $\mathscr{X}$. Then $X_{\overline{\mathbb{K}}} \cong \mathbb{P}^1_{\overline{\mathbb{K}}}$.

Let v_0 be any vertex of G. Let D be a divisor on $\Gamma_{\mathbb{Q}}$. Since G is a tree, D is linearly equivalent to $(\deg D)[v_0]$. It follows that $r_\Gamma(D) = \deg(D)$ if $\deg(D) \geq 0$ and that $r_\Gamma(D) = -1$ if $\deg(D) < 0$. Let $\widetilde{D}$ be any divisor on $X_{\overline{\mathbb{K}}}$ such that $\tau_*(\widetilde{D}) = D$. Then $\deg(\widetilde{D}) = \deg(D)$ (cf. Proposition 2.7(4)). By the Riemann–Roch formula on $\mathbb{P}^1_{\overline{\mathbb{K}}}$, we have $r_X(\widetilde{D}) = \deg(D)$ if $\deg(D) \geq 0$, and $r_X(\widetilde{D}) = -1$ if $\deg(D) < 0$. Thus we get $r_\Gamma(D) = r_X(\widetilde{D})$

Case 2. Suppose that $g(G, \omega) = 1$. In this case, $\omega = \mathbf{0}$, or there exists one vertex v_1 of G with $\omega(v_1) = 1$ and $\omega(v) = 0$ for the other vertices.

Subcase 2-1. Suppose that $w = \mathbf{0}$. Then $g(\Gamma) = 1$. Let D be a divisor on $\Gamma_{\mathbb{Q}}$. As in the Case 1 of the proof of Theorem 8.3, we may assume that D is linearly equivalent to an effective divisor. Also, since the assertion is obvious if $D = 0$, we may assume that $\deg(D) \geq 1$.

We note that if $\deg(D) \geq 2$, then $r_\Gamma(D) \geq 1$. Indeed, let v be any point in Γ. Let D_v be the v-reduced divisor that is linearly equivalent to D. Since $g(G) = 1$, $D_v - D(v)[v]$ is supported at most one point w with $D(w) = 1$. This shows that $D(v) \geq \deg(D) - 1 \geq 1$. Hence $D_v - [v]$ is effective. Since v is arbitrary, we have $r_\Gamma(D) \geq 1$. Repeating this procedure, we obtain $r_\Gamma(D) \geq \deg(D) - 1$. We claim that $r_\Gamma(D) = \deg(D) - 1$. Indeed, if this is not the case, we will then have $\deg(D)[w_1] \sim \deg(D)[w_2]$ for any $w_1, w_2 \in \Gamma$, and thus $g(\Gamma) = 0$, which contradicts $g(\Gamma) = 1$.

Notice that there exists a regular R-curve $\mathscr{X}'$ whose generic fiber X is a smooth connected curve of genus 1 and the special fiber is a geometrically irreducible rational curve with one node. (For example, one takes $\mathscr{X}' = \mathrm{Proj}\left(R[x, y, z]/(y^2z - x^3 - xz^2 - \pi z^3)\right)$, where π is a uniformizer of R.) Then taking a successive blow-ups on the special fiber, we have regular, semi-stable R-curve $\mathscr{X}$ such that the reduction graph is $G = (G, \mathbf{0})$. We write $D = \sum_{i=1}^k n_{v_i}[v_i]$ where $n_{v_i} \geq 0$ for all i. We take $\widetilde{D} = \sum_{i=1}^k n_{v_i} P_i$ such that $\tau(P_i) = v_i$ for $1 \leq i \leq k$. Since $\widetilde{D}$ is effective and $\deg(\widetilde{D}) > 0$, by the Riemann–Roch formula on X, we have $r_X(\widetilde{D}) \geq \deg(\widetilde{D}) - 1$. Hence we obtain $r_\Gamma(D) = r_X(\widetilde{D})$.

Subcase 2-2. Suppose that there exists one vertex v_1 of G with $w(v_1) = 1$ and $\omega(v) = 0$ for the other vertices. Let Γ^ω be the virtual weightless metric graph of (G, ω). Then $g(\Gamma^\omega) = 1$.

As in the Case 1 of the proof of Theorem 8.3, we may assume that D is linearly equivalent to an effective divisor. Also we may assume that $D \neq 0$, so that $\deg(D) \geq 1$. Then the computation in the above subcase gives $r_{(\Gamma,\omega)}(D) = r_{\Gamma^\omega}(D) = \deg(D) - 1$. Let $\mathscr{X}'$ be a regular R-curve whose generic fiber X and the special fiber are both smooth connected curves of genus 1. Then taking a successive blow-ups on the special fiber, we have regular, semi-stable R-curve $\mathscr{X}$ of X such that the reduction graph is (G, w). Then the argument in the above subcase shows that there exists $\widetilde{D} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $\tau_*(\widetilde{D}) = D$ and $r_{(\Gamma,\omega)}(D) = r_X(\widetilde{D})$. □

Next we prove Proposition 1.4.

Proposition (= Proposition 1.4)**.** *Let G be a loopless finite graph. Assume that there exist a complete discrete valuation field $\mathbb{K}$ with the ring of integers R, and a regular, strongly semi-stable,*

totally degenerate R-curve $\mathscr{X}$ with the reduction graph $G = (G, \mathbf{0})$ satisfying the condition (C) *in Question 1.1. Then the Riemann–Roch formula on $\Gamma_{\mathbb{Q}}$ is deduced from the Riemann–Roch formula on $X_{\overline{\mathbb{K}}}$.*

Proof. We take any $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$. By the condition (C), there exists $\widetilde{D} \in \mathrm{Div}(X_{\overline{\mathbb{K}}})$ such that $r_\Gamma(D) = r_X(\widetilde{D})$.

By the Riemann–Roch formula on X, we have

$$r_X(\widetilde{D}) - r_X(K_X - \widetilde{D}) = 1 - g(X) + \deg(\widetilde{D}).$$

Since $\mathscr{X}$ is strongly semi-stable and totally degenerate, we have $g(X) = g(\Gamma)$. By Proposition 2.7, we have $\deg(\widetilde{D}) = \deg D$ and $\tau(K_X) \sim K_G$. Then

$$r_\Gamma(D) - r_X(K_X - \widetilde{D}) = 1 - g(\Gamma) + \deg(D).$$

We put $\widetilde{\mathscr{D}} = \{\widetilde{F} \in \mathrm{Div}(X_{\overline{\mathbb{K}}}) \mid \tau_*(\widetilde{F}) \sim D\}$. By the Riemann–Roch formula on X, we have

$$\max_{\widetilde{F} \in \widetilde{\mathscr{D}}}\{r_X(K_X - \widetilde{F})\} = -1 + g(X) - \deg(\widetilde{D}) + \max_{\widetilde{F} \in \widetilde{\mathscr{D}}}\{r_X(\widetilde{F})\}.$$

Since the right-hand side attains the maximum when $\widetilde{F} = \widetilde{D}$ by Baker's specialization lemma and our choice of $\widetilde{D}$, so does the left-hand side. By the condition (C) and Baker's specialization lemma, the left-hand side is equal to $r_\Gamma(K_\Gamma - D)$. Hence we get $r_X(K_X - \widetilde{D}) = r_\Gamma(K_\Gamma - D)$, and thus

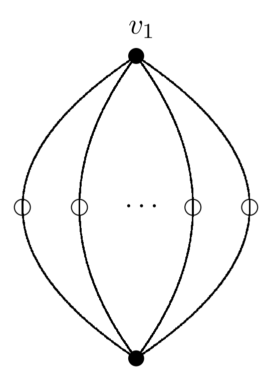
$$r_\Gamma(D) - r_\Gamma(K_\Gamma - D) = 1 - g(\Gamma) + \deg(D)$$

The last equality gives the Riemann–Roch formula on $\Gamma_{\mathbb{Q}}$. □

Remark 8.5. Let G a loopless hyperelliptic graph. Let $\overline{G}$ be the finite graph obtained by contracting all the bridges of G. Let Γ and $\overline{\Gamma}$ be the metric graphs associated to G and $\overline{G}$, respectively. By Theorem 1.2 and Proposition 1.4, the Riemann–Roch formula on $\overline{\Gamma}_{\mathbb{Q}}$ is deduced from the Riemann–Roch formula on a suitable hyperelliptic curve. Since the rank of divisors are preserved under contracting bridges by [6, Corollary 5.11] and [12, Lemma 3.11] (cf. Lemma 2.10), the Riemann–Roch formula on $\Gamma_{\mathbb{Q}}$ is deduced. Since $r_G(D) = r_\Gamma(D)$ for $D \in \mathrm{Div}(G)$ by [18], the the Riemann–Roch formula on G is also deduced.

We finish by giving some examples of ranks of divisors on metric graphs.

Example 8.6. Let G be the following graph of genus $g \geq 3$, where each vertex is given by a white or black circle. Let Γ be the metric graph associated to G. Let $D = [v_1] + [v_2]$. It is easy to see $r_\Gamma(D) = 1$.

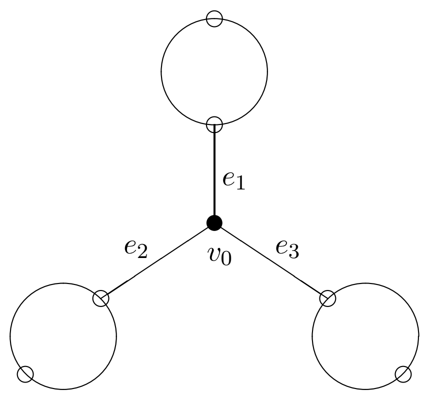


We take a complete valuation field $\mathbb{K}$ with ring integer R and residue field k such that there exists a regular, strongly semi-stable, totally degenerate R-curve $\mathscr{X}$ such that the generic fiber X

is *non-hyperelliptic* and the dual graph of the special fiber is equal to G. There exists such $\mathscr{X}$, see, e.g., [6, Example 3.6].

Let $\widetilde{D}$ be a divisor on $X_{\overline{\mathbb{K}}}$ such that $\tau_*(\widetilde{D}) = D$. Then $\deg(\widetilde{D}) = 2$. Since X is assumed to be non-hyperelliptic, we have $r_X(\widetilde{D}) \neq 1$. It follows that the condition (C) in Question 1.1 is not satisfied for this choice of $\mathscr{X}$. (Indeed, we have to choose a model $\mathscr{X}$ such that X is hyperelliptic to satisfy the condition (C).)

Example 8.7. Let G be the following three petal graph of genus 3, where each vertex is given by a white circle or a black circle. Let Γ be the metric graph associated to G. Let $D = 2[v_0]$. It is easy to see $r_\Gamma(D) = 1$. Thus Γ is a hyperelliptic graph.



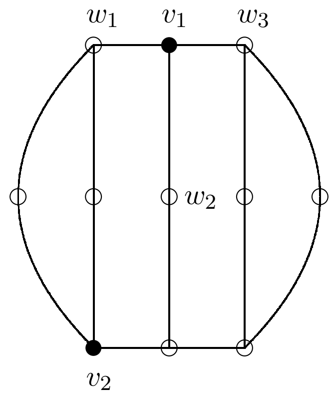
Let $\mathbb{K}$ be a complete valuation field with ring integer R and algebraically residue field k such that $\mathrm{char}(k) \neq 2$. Let $\mathscr{X}$ be a regular, strongly semi-stable, totally degenerate R-curve with the reduction graph G. Let X be the generic fiber of $\mathscr{X}$.

Since the vertex v_0 has three positive-type bridges e_1, e_2, e_3, the graph $G = (G, \mathbf{0})$ does not satisfy the condition (i) in Theorem 1.2. Then Theorem 1.9 tells us that X is not hyperelliptic. The argument in Example 8.6 (which agrees with Theorem 1.2) shows that there exists no divisor $\widetilde{D}$ on $X_{\overline{\mathbb{K}}}$ with $r_X(\widetilde{D}) = 1$ such that $\tau_*(\widetilde{D}) = D$.

Example 8.8. Let G be the following hyperelliptic graph of genus 4, where each vertex is given by a white or black circle. Let Γ be the metric graph associated to G. The involution ι of Γ is given by the reflection relative to the horizontal line through w_2.

Let $D = 3[v_1] + [v_2]$. Taking a function f on Γ such that $f(v_1) = 1$ and $f(w) = 0$ for any $w \in V(G) \smallsetminus \{v_1\}$ and f is linear on each edge. Then $D + (f) = [v_2] + [w_1] + [w_2] + [w_3]$. Since $[v_2] + [w_1] \sim [w] + [\iota(w)]$ for any $w \in \Gamma$ by Lemma 3.9, we have $r_\Gamma(D) \geq 1$. In fact, it is easy to see

$r_\Gamma(D) = 1$.



The graph G has no bridges. Let $\mathbb{K}$ be a complete valuation field with ring integer R and algebraically closed residue field k such that $\mathrm{char}(k) \neq 2$. By Theorem 1.9, we take a regular, strongly semi-stable, totally degenerate R-curve $\mathscr{X}$ with reduction graph $G = (G, \mathbf{0})$ such that the generic fiber X is hyperelliptic. Let ι_X be the involution on X as in Theorem 1.2. As we have shown, this model $\mathscr{X}$ satisfies the condition (C') in Theorem 1.2(iii).

Let $P_1, P_2 \in X(\overline{\mathbb{K}})$ be any points with $\tau(P_1) = v_1$ and $\tau(P_2) = v_2$. Since the Zariski closure of P_i in $\mathscr{X}$ meets only one irreducible component on the special fiber, and since $\iota(v_1) \neq v_2$, we have $\iota_X(P_1) \neq P_2$. We set $\widetilde{D} = 3P_1 + P_2$. By Theorem 7.1, we have $r_X(\widetilde{D}) = 0$. Hence $r_\Gamma(\tau_*(\widetilde{D})) \neq r_X(\widetilde{D})$. This example shows that we need to replace D with a divisor E linearly equivalent to D to satisfy the condition (C') in Theorem 1.2. See Remark 1.7.

Appendix. Deformation theory

Let $\langle\iota\rangle$ denote the finite group of order 2. To prove Theorem 1.9 in §3, we used $\langle\iota\rangle$-equivariant deformation theory on curves and nodes. Since we cannot find a suitable reference in the form we used in §3 (for any characteristic $\neq 2$), we put together necessary results in this appendix. Unlike the previous sections, proofs of the results in this appendix are only sketched. Our basic references are [14] and [23].

A.1. **Equivariant deformation of curves.** In this subsection, we describe the $\langle\iota\rangle$-equivariant deformation theory on curves.

Let k be a field. We assume that $\mathrm{char}(k) \neq 2$. We put

$$\Lambda := \begin{cases} k & \text{if } \mathrm{char}(k) = 0, \\ \text{the ring of Witt vectors over } k & \text{if } \mathrm{char}(k) > 0. \end{cases}$$

Let X_0 be a stable curve of genus g over k. Let A be an Artin local ring with residue field k. A *deformation* of X_0 to A is a stable curve $\mathcal{X} \to \mathrm{Spec}(A)$ with an identification $\mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X_0$. Two deformations $\mathcal{X} \to \mathrm{Spec}(A)$ and $\mathcal{X}' \to \mathrm{Spec}(A)$ are said to be isomorphic if there exists an isomorphism $\mathcal{X} \to \mathcal{X}'$ over A which restricts to the identity on the special fiber X_0.

Let $\mathscr{A}$ be the category of Artin local Λ-algebras with residue field k. The *deformation functor for* X_0 is a functor

$$\mathrm{Def}_{X_0} : \mathscr{A} \to (Sets)$$

that assigns to any $A \in Ob(\mathscr{A})$ the set of isomorphism classes of deformations of X_0 to A.

Suppose now that X_0 is a hyperelliptic stable curve of genus g over k (cf. Definition 4.1). For an Artin local Λ-algebra A with residue field k, An *$\langle\iota\rangle$-equivariant deformation of X_0 to A* is the pair of a stable curve $\mathcal{X} \to \mathrm{Spec}(A)$ with an identification $\mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = X_0$ and an $\langle\iota\rangle$-action on $\mathcal{X}$ whose restriction to the special fiber X_0 is the given $\langle\iota\rangle$-action. Two equivariant deformations

$\mathcal{X} \to \mathrm{Spec}(A)$ and $\mathcal{X}' \to \mathrm{Spec}(A)$ of X_0 are said to be isomorphic if there is an $\langle\iota\rangle$-equivariant isomorphism $\mathcal{X}' \to \mathcal{X}$ over A whose restriction to the special fiber X_0 is the identity.

The *equivariant deformation functor* for X_0 is a functor

$$\mathrm{Def}_{(X_0,\iota)} : \mathscr{A} \to (Sets)$$

which assigns to $A \in Ob(\mathscr{A})$ the set of isomorphism classes of equivariant deformations of X_0 to A. The deformation functor Def_{X_0} has a natural $\langle\iota\rangle$-action. We define $\mathrm{Def}^{\iota}_{X_0}$ to be the subfunctor of Def_{X_0} consisting of the ι-invariant elements of Def_{X_0}. We define a canonical morphism $\mathrm{Def}_{(X_0,\iota)} \to \mathrm{Def}_{X_0}$ by forgetting the $\langle\iota\rangle$-action, which factors through $\mathrm{Def}^{\iota}_{X_0}$.

Lemma A.1. *The canonical morphism* $\mathrm{Def}_{(X_0,\iota)} \to \mathrm{Def}^{\iota}_{X_0}$ *is an isomorphism.*

Proof. One can obtain the assertion by using [14, Theorem 1.11]. □

Proposition A.2. *The functor* $\mathrm{Def}_{(X_0,\iota)}$ *can be pro-represented by a formal power series over* Λ.

Proof. The deformation functor Def_{X_0} is pro-represented by $\mathrm{Spf}\,\Lambda[[t_1,\ldots,t_{3g-3}]]$ by [14]. Since $\mathrm{Def}_{(X_0,\iota)} = \mathrm{Def}^{\iota}_{X_0}$ by Lemma A.1, $\mathrm{Def}_{(X_0,\iota)}$ can be pro-represented by the formal subscheme of $\mathrm{Spf}\,\Lambda[[t_1,\ldots,t_{3g-3}]]$ consisting of "the ι-invariants".

Taking a suitable coordinate, we can express the $\langle\iota\rangle$-action as

$$\iota^*(t_1) = t_1, \ldots, \iota^*(t_s) = t_s, \iota^*(t_{s+1}) = -t_{s+1}, \ldots, \iota^*(t_{3g-3}) = -t_{3g-3},$$

for some $0 \le s \le 3g-3$, since the order 2 of ι is invertible in Λ. It follows that $\mathrm{Def}_{(X_0,\iota)}$ is a formal power series over Λ. □

Remark A.3. The universal equivariant deformation of X_0 is algebraizable since the universal deformation $\mathscr{C} \to \mathrm{Spf}\,\Lambda[[t_1,\ldots,t_{3g-3}]]$ is algebraizable.

A.2. **Equivariant deformation of nodes.** In this subsection, we describe the $\langle\iota\rangle$-equivariant deformation theory on nodes.

Let $\mathcal{O} \cong k[[x,y]]/(xy)$ be a node over k with an $\langle\iota\rangle$-action. An ι-equivariant deformation of $\mathcal{O}$ to A consists of a co-cartesian diagram of local homomorphisms

$$\begin{array}{ccc} \mathcal{O} & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & A \end{array}$$

of A-algebras, where B is a local A-algebra with an $\langle\iota\rangle$-action on B over A which is compatible with the given $\langle\iota\rangle$-action on $\mathcal{O}$.

Let $\mathscr{A}$ be the category of Artin local Λ-algebras with residue field k, as before. The *ι-equivariant deformation functor for $\mathcal{O}$* is the functor

$$\mathrm{Def}_{(\mathcal{O},\iota)} : \mathscr{A} \to (Sets)$$

that assigns to any $A \in Ob(\mathscr{A})$ the set of isomorphism classes of ι-equivariant deformations of $\mathcal{O}$ to A.

We will describe the pro-representable hull of $\mathrm{Def}_{(\mathcal{O},\iota)}$. Let X_0 with an $\langle\iota\rangle$-action be a hyperelliptic semi-stable curve over k, and let c be an ι-fixed smooth point. Then $\mathcal{O} := \widehat{\mathcal{O}_{X_0,c}}$ is a node with an $\langle\iota\rangle$-action. The following lemma describes how $\langle\iota\rangle$ acts on $\mathcal{O}$ concretely.

Lemma A.4. *Let $\mathcal{O}$ be as above with nontrivial ι-action. Then there exists an isomorphism $\mathcal{O} \cong k[[x,y]]/(xy)$ for which the $\langle\iota\rangle$-action on $k[[x,y]]/(xy)$ is given by either one of the following*:

(A.4.1) $$\iota(x) = y, \quad \iota(y) = x,$$

(A.4.2) $$\iota(x) = -x, \quad \iota(y) = -y.$$

Proposition A.5. *Let*

$$\begin{array}{ccc} k[[x,y]]/(xy) & \longleftarrow & \Lambda[[x,y,t]]/(xy-t) \\ \uparrow & & \uparrow \\ k & \longleftarrow & \Lambda[[t]] \end{array} \tag{A.5.1}$$

be the miniversal deformation of $\mathrm{Def}_{k[[x,y]]/(xy)}$.

(Case 1) *Consider an* $\langle\iota\rangle$*-action on* $k[[x,y]]/(xy)$ *as in* (A.4.1). *Let* ι *act on* $\Lambda[[x,y,t]]/(xy-t)$ *by* $\iota(x)=y$ *and* $\iota(y)=x$.

(Case 2) *Consider an* $\langle\iota\rangle$*-action on* $k[[x,y]]/(xy)$ *as in* (A.4.2). *Let* ι *act on* $\Lambda[[x,y,t]]/(xy-t)$ *by* $\iota(x)=-x$ *and* $\iota(y)=-y$.

Then, in each of the above two cases, the diagram (A.5.1) *with the* $\langle\iota\rangle$*-action is a miniversal equivariant deformation of* $k[[x,y]]/(xy)$. *Thus the natural morphism* $h_{\Lambda[[t]]}\to\mathrm{Def}_{(k[[x,y]]/(xy),\iota)}$ *given by the diagram* (A.5.1) *with the* $\langle\iota\rangle$*-action shows us that* $\Lambda[[t]]$ *is a pro-representable hull for* $\mathrm{Def}_{(k[[x,y]]/(xy),\iota)}$ *in each case.*

Proof. We sketch the proof of Case (1). Case (2) is shown similarly. One can show the following claim (here we use $\mathrm{char}(k)\neq 2$).

Claim A.5.2. *Let A be an Artin local ring with maximal ideal m, and let I be an ideal of A with $mI=0$.*[1] *Then*

$$h_{\Lambda[[t]]}(A)\to\mathrm{Def}_{(\mathcal{O},\iota)}(A)\times_{\mathrm{Def}_{(\mathcal{O},\iota)}(A/I)} h_{\Lambda[[t]]}(A/I)$$

is surjective.

By [23, Remark 2.3], Claim A.5.2 tells us that $h_{\Lambda[[t]]}\to\mathrm{Def}_{(k[[x,y]]/(xy),\iota)}$ is smooth. In order to complete the proof, we need show the map

$$h_{\Lambda[[t]]}(k[\epsilon]/(\epsilon^2))\to\mathrm{Def}_{(k[[x,y]]/(xy),\iota)}(k[\epsilon]/(\epsilon^2))$$

is injective. However it is rather trivial since we have a commutative diagram

$$\begin{array}{ccc} h_{\Lambda[[t]]}(k[\epsilon]/(\epsilon^2)) & \longrightarrow & \mathrm{Def}_{(k[[x,y]]/(xy),\iota)}(k[\epsilon]/(\epsilon^2)) \\ \| & & \downarrow \\ h_{\Lambda[[t]]}(k[\epsilon]/(\epsilon^2)) & \xrightarrow{\cong} & \mathrm{Def}_{k[[x,y]]/(xy)}(k[\epsilon]/(\epsilon^2)), \end{array}$$

where the bijectivity of the bottom row follows from the fact that $\Lambda[[t]]$ is a hull for $\mathrm{Def}_{k[[x,y]]/(xy)}$. Thus we have shown that it is a hull for $\mathrm{Def}_{\mathcal{O}}$ in this case. □

Remark A.6. (1) The canonical morphism $\mathrm{Def}_{(\mathcal{O},\iota)}\to\mathrm{Def}_{\mathcal{O}}$ is smooth since so are $h_{\Lambda[[t]]}\to\mathrm{Def}_{\mathcal{O}}$ and $h_{\Lambda[[t]]}\to\mathrm{Def}_{(\mathcal{O},\iota)}$.

(2) We have isomorphisms

$$h_{\Lambda[[t]]}(k[\epsilon]/(\epsilon^2))\cong\mathrm{Def}_{(X_0,\iota)}(k[\epsilon]/(\epsilon^2))\cong\mathrm{Def}_{X_0}(k[\epsilon]/(\epsilon^2))$$

since $\Lambda[[t]]$ is a hull for the deformation functors.

For a functor $F:\mathscr{A}\to(Sets)$, we define $\widehat{F}(R):=\varprojlim_i F(R/\pi^i)$. The morphism $\mathrm{Def}_{(\mathcal{O},\iota)}\to\mathrm{Def}_{\mathcal{O}}$ induces $\widehat{\mathrm{Def}_{(\mathcal{O},\iota)}}(R)\to\widehat{\mathrm{Def}_{\mathcal{O}}}(R)$.

Corollary A.7. *The map* $\widehat{\mathrm{Def}_{(\mathcal{O},\iota)}}(R)\to\widehat{\mathrm{Def}_{\mathcal{O}}}(R)$ *is surjective.*

Proof. The assertion follows from Remark A.6 and [23, Remark 2.4]. □

[1] In other words, the canonical homomorphism $A\to A/I$ is a small extension.

A.3. **Global-local morphism.** Let X_0 be a stable curve of genus g over k, and let $p_1, \dots p_t$ be all the nodes of X_0. We assume that any node is defined over k. Then, one can define a morphism

$$\Phi^{gl} : \mathrm{Def}_{X_0} \to \prod_{i=1}^{t} \mathrm{Def}_{p_i}$$

that assigns to any deformation $\mathcal{X} \to \mathrm{Spec}(A)$ of X_0 the deformation $A \to \widehat{\mathcal{O}_{\mathcal{X},p_i}}$ of each node $\widehat{\mathcal{O}_{X_0,p_i}}$. We call Φ^{gl} the global-local morphism.

Assume now that X_0 a hyperelliptic stable curve over k with hyperelliptic involution ι. Let $p_1, \dots, p_r$ be the nodes of X_0 fixed by ι, and let $p_{r+1}, \dots, p_s$ be nodes such that $p_{r+1}, \dots, p_{r+s}$, $\iota(p_{r+1}), \dots, \iota(p_{r+s})$ are the distinct nodes that are not fixed by ι. One can also define a natural morphism

$$\Phi_\iota^{gl} : \mathrm{Def}_{(X_0,\rho_0)} \to \prod_{i=1}^{r} \mathrm{Def}_{(p_i,\iota)} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i} \tag{A.7.1}$$

that assigns to any deformation $\mathcal{X} \to \mathrm{Spec}(A)$ of X_0 the $\langle\iota\rangle$-equivariant deformation $A \to \widehat{\mathcal{O}_{\mathcal{X},p_i}}$ of the node $\widehat{\mathcal{O}_{X_0,p_i}}$ for $1 \le i \le r$ and the deformation $A \to \widehat{\mathcal{O}_{\mathcal{X},p_i}}$ of the node $\widehat{\mathcal{O}_{X_0,p_i}}$ for $r+1 \le i \le r+s$. We also call Φ^{gl} the global-local morphism.

Proposition A.8. *The morphism Φ_ι^{gl} is smooth.*

Proof. Using the natural morphism $\iota_* : \mathrm{Def}_{p_i} \to \mathrm{Def}_{\iota(p_i)}$, we obtain an $\langle\iota\rangle$-action on

$$\prod_{i=1}^{r} \mathrm{Def}_{p_i} \times \prod_{i=r+1}^{r+s} (\mathrm{Def}_{p_i} \times \mathrm{Def}_{\iota(p_i)}).$$

We define the morphism

$$\Psi : \prod_{i=1}^{r} \mathrm{Def}_{(p_i,\iota)} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i} \to \prod_{i=1}^{r} \mathrm{Def}_{p_i} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{\iota(p_i)} \tag{A.8.1}$$

by the products of the canonical morphisms $\mathrm{Def}_{(p_i,\iota)} \to \mathrm{Def}_{p_i}$ for $i = 1, \dots, r$ and the graph embeddings $\mathrm{Def}_{p_i} \to \mathrm{Def}_{p_i} \times \mathrm{Def}_{\iota(p_i)}$ of ι_* for $i = r+1, \dots, r+s$. Then, for $i = r+1, \dots, r+s$, the morphism $\mathrm{Def}_{p_i} \to \mathrm{Def}_{p_i} \times \mathrm{Def}_{\iota(p_i)}$ is an isomorphism onto the subfunctor of $\mathrm{Def}_{p_i} \times \mathrm{Def}_{\iota(p_i)}$ consisting of the ι-invariant elements. Further, Remark A.6 tells us that the map $\Psi(k[\epsilon]/(\epsilon^2))$ between the tangent spaces is an isomorphism onto the set of ι-invariants of

$$\left(\prod_{i=1}^{r} \mathrm{Def}_{p_i} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{\iota(p_i)}\right)(k[\epsilon]/(\epsilon^2)). \tag{A.8.2}$$

Thus the tangent space $\left(\prod_{i=1}^{r} \mathrm{Def}_{(p_i,\iota)} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i}\right)(k[\epsilon]/(\epsilon^2))$ is identified with the set of ι-invariants of (A.8.2). Through this identification, $\Phi_\iota^{gl}(k[\epsilon]/(\epsilon^2))$ is regarded as the restriction of $\Phi^{gl}(k[\epsilon]/(\epsilon^2))$ to the ι-invariants. By [14, Theorem 1.11], Φ^{gl} induces a surjective map between tangent spaces. Since 2 is invertible in k, the induced map between ι-invariants is also surjective, so that $\Phi_\iota^{gl}(k[\epsilon]/(\epsilon^2))$ is surjective.

On the other hand, Propositions A.2 and A.5 say that the pro-representable hulls of $\mathrm{Def}_{(X_0,\iota)}$ and $\prod_{i=1}^{r} \mathrm{Def}_{(p_i,\iota)} \times \prod_{i=r+1}^{r+s} \mathrm{Def}_{p_i}$ are both formal power series over Λ. Therefore, the surjectivity of the map $\Phi_\iota^{gl}(k[\epsilon]/(\epsilon^2))$ between tangent spaces implies that Φ_ι^{gl} is smooth. $\square$

For a functor $F : \mathscr{A} \to (Sets)$, we define $\widehat{F}(R) := \varprojlim_i F(R/\pi^i)$. Our Φ^{gl} induces a map

$$\widehat{\Phi^{gl}}(R) : \widehat{\mathrm{Def}_{(X_0,\iota)}}(R) \to \left(\prod_{i=1}^{r} \widehat{\mathrm{Def}_{(p_i,\iota)}} \times \prod_{i=r+1}^{r+s} \widehat{\mathrm{Def}_{p_i}}\right)(R). \tag{A.8.3}$$

Corollary A.9. *$\widehat{\Phi^{gl}}(R)$ is surjective.*

Proof. The assertion follows from Proposition A.8 and [23, Remark 2.4]. □

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